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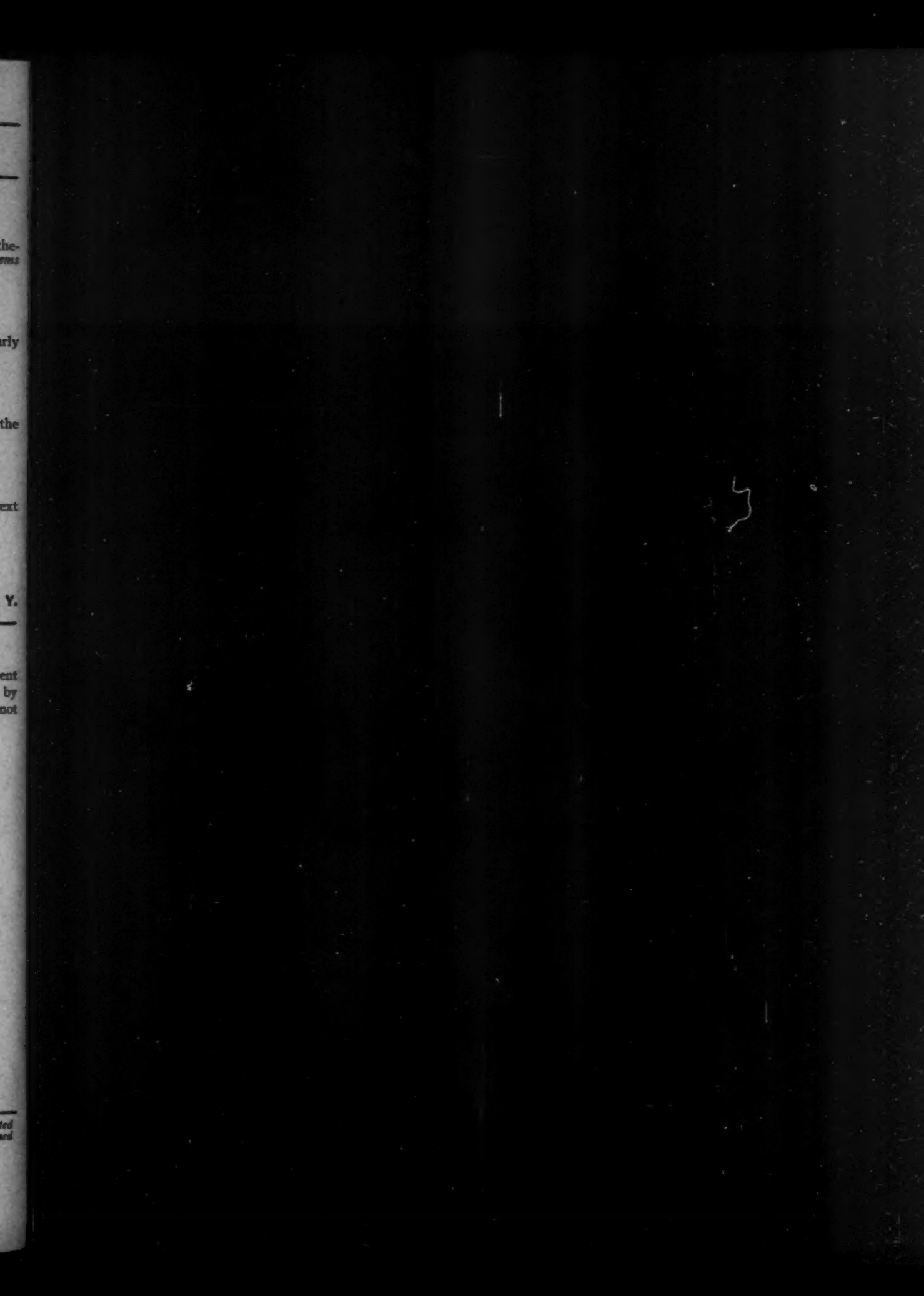

Mathematical Reviews

Vol. 9, No. 8 SEPTEMBER, 1948 Pages 403-482

FOUNDATIONS

Markov, A. On the representation of recursive functions. Doklady Akad. Nauk SSSR (N.S.) **58**, 1891–1892 (1947). (Russian)

S. C. Kleene has proved (A) that every general recursive function of n variables can be represented in the form

$$(1) \qquad P(\mu y \cdot Q(x_1, \cdots, x_n, y) = 0),$$

where P and Q are primitive recursive, and $\mu y \cdot \mathbf{X}$ is the least natural number y such that $\mathbf{X}$ holds, or 0 if there is none [Math. Ann. **112**, 727–742 (1936)]; and (B) that a fixed "universal" primitive recursive function P can be found, such that all general recursive functions are obtained by suitable choice of primitive recursive Q in (1) [Trans. Amer. Math. Soc. **53**, 41–73 (1943); these Rev. **4**, 126]. In the present note the theorem is announced [without proof] that a necessary and sufficient condition for a primitive recursive function $P(x)$ to be universal, in the above sense, is that it takes every natural number as value an infinity of times.

M. H. A. Newman (Manchester).

Aubert, K. E. A group-theoretical remark of E. Hoff-Hansen concerning certain expressions in the quantification theory. Arch. Math. Naturvid. **49**, no. 7, 151–156 (1947).

This paper had its origin in posthumous notes left by E. Hoff-Hansen. It is first shown that logical expressions containing quantifiers form an Abelian group with respect to a certain rule of combination $A \circ B$. A condition stated in terms of this group is then shown to be necessary and sufficient for the logical equivalence of two expressions.

O. Frink (State College, Pa.).

Wang, Hao. A note on Quine's principles of quantification. J. Symbolic Logic **12**, 130–132 (1947).

The author shows that Quine's six principles of quantification *100–*105 [see W. V. Quine, Mathematical Logic, Norton, New York, 1940; these Rev. **2**, 65] can be replaced by only four similar principles, the last of which is a strengthened form of *105. He uses the altered definition of closure due to G. D. W. Berry. *O. Frink* (State College, Pa.).

Fitch, Frederic B. Corrections to two papers on modal logic. J. Symbolic Logic **13**, 38–39 (1948).

The papers appeared in the same J. **2**, 125–128 (1937); **4**, 115–116 (1939); these Rev. **1**, 131.

Beth, E. W. The origin and growth of symbolic logic. Synthèse **6**, 268–274 (1948).

Heyting, Arend. Formal logic and mathematics. Synthèse **6**, 275–282 (1948).

Feys, Robert. Logique formalisée et philosophie. Synthèse **6**, 283–298 (1948).

Lectures on the occasion of the Boole-de Morgan centenary.

Gödel, Kurt. What is Cantor's continuum problem? Amer. Math. Monthly **54**, 515–525 (1947).

This is an exposition of the continuum problem. It is elementary in the sense that it is written in a nontechnical language and that most of it can be understood without an extensive knowledge of the foundations of set theory; yet it ought to be of interest even to the specialist, in particular the parts where the author discusses the possible future developments of the problem. The continuum problem can be stated as follows: how many points are there on a straight line in Euclidean space? An equivalent formulation is: how many sets of integers do there exist? Stated in terms of cardinal numbers, Cantor's continuum hypothesis is: $2^{\aleph_0} = \aleph_1$. It is natural to generalize this and to conjecture that, for any ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. This is the generalized continuum hypothesis.

The problem is important among other things because the evaluation of infinite cardinal products depends to a large extent on its solution. It is, therefore, surprising that relatively little is known about it in spite of the fact that it was formulated over sixty years ago. This suggests that the difficulties may not be of a purely mathematical nature and that the solution of this and related problems may require "a more profound analysis (than mathematics is accustomed to give) of the meaning of the terms occurring in them (such as 'set,' 'one-to-one correspondence,' etc.) and of the axioms underlying their use." Of the analyses that have been proposed so far, the author rejects Brouwer's intuitionism and "the 'half-intuitionistic' standpoint along the lines of H. Poincaré and H. Weyl" but favors an interpretation of the problems within a system of axioms for set theory such as can be found in the works of P. Bernays and J. von Neumann. Thus interpreted the continuum hypothesis may be either demonstrable or disprovable or undecidable. Although the only known result is that Cantor's conjecture is not disprovable, the author believes that the third possibility is most likely.

Suppose we succeed in proving that the continuum hypothesis is undecidable, is the question thereby settled? In the author's opinion, it is not, for he believes that the axioms of set theory "describe some well-determined reality" and that Cantor's conjecture must therefore be either true or false. Its undecidability would therefore merely show that the present axioms are not sufficient. Our next task would then be to try to discover new axioms which would make the problem decidable. Various possibilities are discussed, but the author's final conclusion is that the new axioms will probably be such that they will make it possible to disprove Cantor's conjecture. His main reasons for believing this are that the continuum hypothesis has many unexpected and implausible consequences in point-set theory while numerous plausible propositions are known which imply its negation. *B. Jónsson* (Providence, R. I.).

Riabouchinsky, Dimitri. Sur le concept de l'origine d'un nombre et le problème du continu. C. R. Acad. Sci. Paris **225**, 552–554 (1947).

The author distinguishes somewhat obscurely between the "origine" of a number and its "valeur numérique." As Church has pointed out [J. Symbolic Logic **13**, 55 (1948)], Riabouchinsky appears to have in mind the semantical distinction (due to Frege) between the meaning and the denotation of a numerical expression. Riabouchinsky suggests that two numbers are not to be regarded as identical unless their "origines" are the same as well as their "valeurs numériques." Careful criteria for this very strong kind of identity are not given. Possibly what is intended is something close to Carnap's *L*-equivalence [see R. Carnap, Meaning and Necessity, University of Chicago Press, 1947, passim]. The author does not make clear what connection his distinction has with the continuum problem.

R. M. Martin (Philadelphia, Pa.).

Pompeiu, D. Sur l'arithmétique de l'infini. Acad. Roum. Bull. Sect. Sci. **24**, 569–572 (1943).

Schröter, K. Die Arithmetik der natürlichen Zahlen im Rahmen der Theorie der Verbände. Math. Ann. **120**, 197–201 (1948).

The author characterizes the natural numbers axiomatically in terms of lattice theory as a linearly ordered system with a least element 0, satisfying the descending chain condition, such that for each element there is an immediately greater element and for each element (except 0) there is an immediately lesser element. This characterization concerns only the order-relationship of the natural numbers, not addition and multiplication. It is stated without proof that this axiom system is semantically complete. It is not clear to the reviewer whether the author intends to imply that this follows from known results on other characterizations of the system, or that the author has a direct proof. Some consequences in the realm of mathematical logic are discussed. *P. M. Whitman* (Silver Spring, Md.).

Goodstein, R. L. The strong convergence of the exponential function. J. London Math. Soc. **22** (1947), 200–205 (1948).

A sequence of rational numbers s_n is said to be strongly convergent in the scale $r \geqq 2$ in case there exists a recursive function $n(k)$ such that, for all natural numbers k and all $n \geqq n(k)$, $[r^k s_n] = [r^k s_{n(k)}]$. The author points out that the concept is of interest, not only from the foundational viewpoint, but also for computational purposes. It is shown that a sufficient condition for the strong convergence of a sequence s_n is the existence of recursive functions $n(k)$, $i(p, q)$ and $N(p, q)$ such that, for all integral $k > 0$ and $n \geqq n(k)$, $|s_n - s_{n(k)}| < 1/k$, and for all integral $p, q \neq 0$, and $n \geqq N(p, q)$, $|s_n - p/q| > 1/i(p, q)$. Using this criterion, the author shows that the nth partial sums of the Maclaurin expansion of e^x for any rational x form a strongly convergent sequence. This allows a constructive interpretation of the existence of e^x for rational x. The method of proof affords a bound on the number of successive digits which are the same in any part of the r-ary expansion of e^x for a given rational x. *D. Nelson* (Washington, D. C.).

*** Defrise, Pierre. Visages de la Mathématique.** Office de Publicité, Bruxelles, 1948. 126 pp. 60 francs.

This is a brief presentation of some aspects of mathematics, dealing chiefly with geometry. The axiomatic method, invariants, transformations and groups form the subjects of the three chapters. It is intended to give the interested layman some insight into the basic methods and points of view of modern mathematics. Like many such enterprises it requires some mathematical maturity on the part of the reader if he is to realize the author's intention.

A. Dresden (Swarthmore, Pa.).

Bouligand, Georges. Problèmes. Synthèse globale. C. R. Acad. Sci. Paris **226**, 294–297 (1948).

Continuant les considérations exposées en deux notes précédentes [mêmes C. R. **224**, 1747–1749 (1947); **225**, 780–782 (1947); ces Rev. **9**, 262, 263] l'auteur revient sur les problèmes dont la formulation précise a progressivement requis dans S_i, au delà des processus finis, la suite N des entiers, le continu linéaire L et les divers champs fonctionnels. Il cite B. Levi qui voit dans ces systèmes des ensembles primitifs correspondants à des domaines déductifs dont l'un constitue l'extension naturelle de l'autre. Chaque énoncé livré à S_i par un groupement de problèmes, devient une souche disponible à des énoncés particuliers issus d'un sous-groupement et atteignant un niveau de vérité meilleur. L'idée des niveaux de vérité est donc appuyée sur la théorie de B. Levi. Cette conception pourrait conduire à un intuitionisme, que l'auteur juge, cependant, moins fécond que l'analyse classique. *E. W. Beth* (Amsterdam).

Sakellariou, Nilos. On the foundations of the mathematical science. Bull. Soc. Math. Grèce **19**, 153–180 (1948). (Greek. English summary)

Burington, Richard Stevens. The role of the concept of equivalence in the study of physical and mathematical systems. J. Washington Acad. Sci. **38**, 11 pp. (1948).

Finzi, Bruno. Equilibrio. Rend. Sem. Mat. Fis. Milano **16**, 16–47 (1942).

The purpose of this paper is to review the concept of equilibrium in various fields of science. Starting with elementary statical systems, the author proceeds through the principle of virtual work to the equilibrium of continuous systems. Thermodynamic, chemical and electrostatic equilibrium are discussed on the basis of stationary principles and there is also discussion of equilibrium in economics and biology. *J. L. Synge* (Dublin).

ALGEBRA

Kerawala, S. M. The asymptotic number of three-deep Latin rectangles. Bull. Calcutta Math. Soc. **39**, 71–72 (1947).

The instance $k=3$ of the Erdös-Kaplansky formula [Amer. J. Math. **68**, 230–236 (1946); these Rev. **7**, 407] for the asymptotic numbers of $n \times k$ Latin rectangles is extended to the length of 11 terms in the form $(n!)^{-3}e^3 f(n, 3) = 1 + \sum_1 a_k/k!n^k$ with $a_1 = a_2 = -1$, $a_3 = 2$, $a_4 = 49$, etc. The numbers a_k are obtained by substitution in a recurrence relation previously given by the author [same Bull. **33**, 119–127 (1941); these Rev. **4**, 69]. [Reviewer's note: it seems likely that the coefficients in the

more natural form [cf. Erdös and Kaplansky, loc. cit., equation 19] $1+\sum b_k/k!(n)_k$, $(n)_k=n(n-1)\cdots(n-k+1)$, would be simpler.] *J. Riordan* (New York, N. Y.).

Kerawala, S. M. Asymptotic solution of the "problème des ménages." Bull. Calcutta Math. Soc. **39**, 82–84 (1947).

An 11-term asymptotic series is given for reduced ménage numbers, i.e., for the number of permutations of n elements discordant with the two permutations $12\cdots n$ and $23\cdots n1$. This is of the form $(n!)^{-1}e^2a(n)=1+\sum_1 D_r/r!n^r$ and is equivalent to $(n!)^{-1}e^2a(n)=1+\sum_1(-)^r/r!(n-1)_r$, $(n-1)_r=(n-1)(n-2)\cdots(n-r)$, given by Kaplansky and Riordan [Scripta Math. **12**, 113–124 (1946); these Rev. **8**, 365]. Indeed $D_r=\sum_1^r(-)^iS(i,r)r!/i!$, with $S(i,r)$ a Stirling number of the second kind. *J. Riordan.*

Radhakrishna Rao, C. General methods of analysis for incomplete block designs. J. Amer. Statist. Assoc. **42**, 541–561 (1947).

This is an expository paper elucidating the methods for the analysis of the designs mentioned in the title. Formulae for the analysis of variance with and without the use of interblock estimates are given. The treatment of designs with missing and mixed up plots is also discussed. The author distinguishes two fundamental types: partially balanced incomplete blocks and intra and intergroup balanced incomplete blocks. Every incomplete block design, at present in use, is of one of the two types. It is therefore sufficient to discuss, as the author does, the analysis of these two very general designs. The author gives only the method of analysis and does not present the underlying theory.

H. B. Mann (Columbus, Ohio).

Dinghas, Alexander. Some identities between arithmetic means and the other elementary symmetric functions of n numbers. Math. Ann. **120**, 154–157 (1948).

A proof by induction of the following identities (A_k and G_k are arithmetic and geometric means, respectively, of the first k of $a_1, a_2, \cdots, a_n$):

$$n(A_n-G_n)=\sum_2^n(a_k^1-G_{k-1}^1)^2P_{k-2}(a_k^1, G_{k-1}^1),$$

$$A_n^n-G_n^n=\sum_2^n(a_k-A_{k-1})^2k^{-1}P_{k-2}(A_k, A_{k-1})a_{k+1}\cdots a_n,$$

where $kl=1$, $P_0=1$, $P_n(x,a)=xP_{n-1}(x,a)+(n+1)a^n$. The second is used to derive identities containing means of elementary symmetric functions. *J. Riordan.*

Petr, K. On a theorem about the coefficients of symmetric functions. Rozpravy II. Třídy České Akad. **52**, no. 11, 10 pp. (1942). (Czech)

Petr, K. On the use of an identity in the theory of symmetric functions. Rozpravy II. Třídy České Akad. **52**, no. 12, 20 pp. (1942). (Czech)

German translations appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **43**, 132–143, 144–162 (1942); these Rev. **8**, 431.

Habicht, Walter. Eine Verallgemeinerung des Sturmschen Wurzelzählverfahrens. Comment. Math. Helv. **21**, 99–116 (1948).

Let $f=f_{n+1}(x)$ and $g=g_n(x)$ be two polynomials with indeterminate coefficients u_i and v_j and let Γ be the ring of integers. Polynomials p_r and q_r of degrees $n-r-1$ and $n-r$, respectively, are determined such that $f_r=p_rf+q_rg$ is of degree r, and the coefficients of p_r, q_r and f_r are in the ring $\Gamma[u_i, v_j]$. Then $f, g, f_{n-1}, \cdots, f_0$ form the generalized Sturmian chain. Specialize the indeterminate coefficients to be elements from an arbitrary field K. Call f_r complete if it is of degree r, and defective if it is of degree $s<r$. If f_{r+1} is complete and f_r is defective of degree s, then $f_{r-1}, \cdots, f_{s+1}$ are identically zero and f_s is complete and proportional to f_r.

Now assume K to be an ordered field and let $w(\xi)$ be the number of changes of sign in the chain for $x=\xi$, an element of K. Call ξ critical if $f(\xi)=0$. Then if ξ_0 is not critical there is a deleted neighborhood $U(\xi_0)$ for which $w(\xi)$ is constant for ξ in $U(\xi_0)$. Finally suppose that K is real and closed. Let $j(\xi, f, (a_s, a_s))\equiv\operatorname{sgn} f(b_s)-\operatorname{sgn} f(b_s)/2$, where $a_s\leqq b_s<\xi<b_s\leqq a_s$, ξ is a critical point, and b_s and b_s such that ξ is the only critical point in (b_s, b_s). Let f and g have no common roots in (a_s, a_s), and let f be different from zero at the end points. Then if $\xi_1, \cdots, \xi_m$ are the critical points of f in (a_s, a_s), $w(a_s)-w(a_s)=\sum_{k=1}^m j(\xi_k, f, (a_s, a_s))\operatorname{sgn} g(\xi_k)$. This contains Sturm's theorem as a special case.

A. W. Goodman (New Brunswick, N. J.).

Opitz, G. Praktische Verfahren zur Lösung von Gleichungen vierten Grades. Z. Angew. Math. Mech. **25/27**, 171–173 (1947).

Describes the solution of equations of the fourth degree in one unknown by means of cubic resolvents, e.g., by Ferrari's resolvent. The author determines the conditions that for the chosen root of one of the cubic resolvents the functions x_1+x_2, x_3+x_4, $(x_1-x_2)^2$, $(x_3-x_4)^2$, x_1x_2, x_3x_4 of the roots are all real. *E. Bodewig* (The Hague).

Volnina, N. V. On the reducibility of polynomials in irrational fields. Doklady Akad. Nauk SSSR (N.S.) **58**, 1873–1875 (1947). (Russian)

Let K be the field of rationals and let $f(x,\beta)$ be a polynomial with coefficients in $K(\beta)$, where β is an algebraic number. A practical method for decomposing the polynomial $f(x,\beta)$ into irreducible factors in $K(\beta)[x]$ is given by the author, avoiding the difficulty presented by the existence of infinitely many units in rings of algebraic integers. Let $F(x)$ be the norm of $f(x,\beta)$ over K. If $F(x)=H(x)\cdot R(x)$ and $(H,R)=1$, then one obtains immediately a nontrivial factorization of $f(x,\beta)$ by considering the highest common divisors of $f(x,\beta)$ with H and R, respectively. Hence after a finite number of steps the problem is reduced to the case in which $F(x)$ is the power of an irreducible polynomial $H(x)$ in $K[x]$ and to the question of decomposing $H(x)$ into irreducible factors in $K(\beta)[x]$. It is therefore permissible to assume that the original polynomial f has coefficients in K, and that $f(x,\beta)=f(x)$ and is irreducible in $K[x]$. Let $\alpha_1, \cdots, \alpha_n$ be the roots of $f(x)$ and let $\beta_1, \cdots, \beta_m$ be the conjugates of β over K, where $\beta_1=\beta$. Fix a polynomial $\varphi(x,y)$ in $K[x,y]$ (a suitable linear form will suffice) satisfying the condition that the mn numbers $\gamma_{sr}=\varphi(\alpha_s, \beta_r)$ are distinct. Let $S(z)$ be the polynomial in $K[z]$ whose roots are the γ_{sr} and let $S(z)=s_1(z)\cdots s_e(z)$ be the decomposition of $S(z)$ into irreducible factors in $K[z]$. Let $f_i(x,\beta)$ be the highest common divisor of $f(x)$ and $s_i(\varphi(x,\beta))$. Following a method used by M. Bauer [J. Reine Angew. Math. **163**, 249–250 (1930)] it is proved that $f(x)=f_1(x,\beta)\cdots f_e(x,\beta)$ is the desired decomposition of $f(x)$ into irreducible factors. *O. Zariski.*

Tietze, Heinrich. Verallgemeinerung eines Hamilton-Cayley-Frobeniusschen Satzes auf ein beliebiges Paar vertauschbarer Matrizen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1945/46, 45–56 (1947).

The principal theorem is a special case of a theorem of H. B. Phillips [Amer. J. Math. 41, 266–278 (1919)].

C. C. MacDuffee (Madison, Wis.).

Kofink, W. Zur Diracschen Theorie des Elektrons. V. Über die Paulische Matrix B. Ann. Physik (6) 1, 133–136 (1947).

If each of the 5 linearly independent 4-rowed matrices γ^μ such that

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\delta_{\mu\nu}, \quad \gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4, \qquad \mu,\nu=1,\cdots,5,$$

is assumed to be symmetric or skew, then exactly 3 of them must be symmetric and 2 must be skew. The Pauli (skew) matrix B such that $B\gamma^\mu B^{-1}$ is the transpose of γ^μ for $\mu=1,\cdots,5$ is i times the product of the two skew matrices.

C. C. MacDuffee (Madison, Wis.).

Jonesco, D. V. Un théorème sur les formes quadratiques. Acad. Roum. Bull. Sect. Sci. 27, 176–179 (1947).

Let F, G, H be three quadratic forms of respective discriminants $A=|a_{ik}|$, B, C. Let $h_i=c_{i1}x_1+\cdots+c_{in}x_n$, and φ_j be obtained from B by replacing the jth column by the h's. Let $\Phi(x_1,\cdots,x_n)=\sum a_{ik}\varphi_i\varphi_k$. The discriminant of $\Phi(x_1,\cdots,x_{n-1},0)$ is $B^{2n-4}C^2\xi_n^2AF(\eta_1,\cdots,\eta_n)/H^2(\xi_1,\cdots,\xi_n)$, where $(\xi_1,\cdots,\xi_n)$ is the center I of $H=0$ in $n-1$ dimensions and $(\eta_1,\cdots,\eta_n)$ is the pole with respect to $F=0$ of the polar hyperplane of I with respect to $G=0$. For $n=3$ there is an application.

C. C. MacDuffee (Madison, Wis.).

Edge, W. L. The discriminant of a certain ternary quartic. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 268–272 (1948).

The author shows by actual calculation that the discriminant D of the quartic

$$ax_1^4+bx_2^4+cx_3^4+6fx_2^2x_3^2+6gx_3^2x_1^2+6hx_1^2x_2^2$$

is $abcA^2B^2C^2\Delta^4$, where

$$\Delta=\begin{vmatrix} a & 3h & 3g \\ 3h & b & 3f \\ 3g & 3f & c \end{vmatrix}$$

and A, B and C are the cofactors of a, b and c, respectively. In calculating D the author follows a procedure attributed to Gordan by Klein [F. Klein, Math. Ann. 36, 1–83 (1890)] in which D is obtained as a determinant of order 15.

J. Williamson (Flushing, N. Y.).

Abstract Algebra

***Bourbaki, N. Éléments de mathématique. VI. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre II: Algèbre linéaire.** Actualités Sci. Ind., no. 1032. Hermann et Cie., Paris, 1947. ii+132+4 pp.

The author of these volumes has set himself the task of presenting a systematic and well-balanced exposition of the whole of mathematics, and has properly chosen to achieve the necessary coherence by using a uniform terminology and a conceptual approach, in which special facts and illustrative examples are to be arranged under the appropriate general notions. His treatment of linear algebras follows these principles; bases and matrices are subordinated to left modules and linear mappings. One often needs these concepts in cases when the multiplication of scalars is not commutative; this appropriate measure of generality is here systematically carried out. One regrets only the author's decision to use the well established term "field" ("corps") to include the case of a skew field, in defiance of the fact that much of nonlinear algebra still needs the distinction.

After the preliminaries for left modules over a ring and left vector spaces over a skew field there is a masterful presentation of the dual of a module, with application to vector spaces and to the solution of systems of linear equations with coefficients in a skew field K. The familiar facts for the case of coefficients in a skew subfield K_0 are presented in terms of a novel study of the restriction of the skew field of scalars of a vector space; this study yields also a Galois correspondence between the skew subfields of finite index and certain rings of endomorphisms of the additive group of K. A matrix is defined as a family $a_{\lambda,\mu}$ with values a in any set (which rapidly becomes a ring) and with indices λ and μ running over two finite sets L and M. This device offers certain conceptual advantages (in that L and M can denote the bases of two vector spaces), but is essentially equivalent to the customary $\lambda=1,\cdots,m$, $\mu=1,\cdots,n$. The connection between matrices and linear transformations and the equivalence and similarity of matrices are discussed. The computational aspects of matrix theory (e.g., multiplication by blocks) are presented fully, but without especial emphasis. The last section deals with elementary facts on linear algebras (multiplication tables, ideals, direct products, and sums) and ends with an excellent array of examples of linear algebra (including group algebras and a careful discussion of the relation between left modules and groups with operators). By further terminological larceny the author makes it an "abuse of language" to speak of algebras over a field (he prefers them over a commutative ring).

An appendix treats semi-linear transformations. There is a wealth of well chosen exercises, which should greatly aid the reader in understanding the conceptual approach to linear algebras.

S. MacLane (Chicago, Ill.).

Croisot, Robert. Une interprétation des relations d'équivalence dans un ensemble. C. R. Acad. Sci. Paris 226, 616–617 (1948).

Groupe partiel: c'est un ensemble muni d'une loi satisfaisant aux trois premiers axiomes de Brandt [Math. Ann. 96, 360–366 (1926)] (non toujours définie, associative, unités à droite, à gauche, inverse). Sous groupe partiel (contenant toutes les unités). Les sous groupes partiels forment un treillis. Soit R une relation d'équivalence dans E et $C_R\subset E\times E$; $(x,y)\in C_R$ si $x\equiv y(R)$. Par rapport à la loi $(x,y)(y,z)=(x,z)$, C_R est un groupe partiel. La correspondance $R\to$groupe partiel C_R respecte intersection, produit complété. Si R et R' sont associables C_R et $C_{R'}$ sont permutables ($AB=BA$). Généralisation à la presque associabilité, à la presque permutabilité ($ABA\supset BAB$).

J. Kuntzmann (Grenoble).

Croisot, Robert. Condition suffisante pour l'égalité des longueurs de deux chaînes de mêmes extrémités dans une structure. Application aux relations d'équivalence et aux sous-groupes. C. R. Acad. Sci. Paris 226, 767–768 (1948).

Conditions are stated which assure that two finite chains of elements in a lattice have equal length. These are then applied to lattices of partial subgroups of a partial group,

recently studied by the author [cf. the preceding review], and hence to lattices of equivalence relations, where the conditions are formulated in terms of the "almost associability" of certain relations. Details of the proofs are to appear subsequently. *P. M. Whitman* (Silver Spring, Md.).

Châtelet, Albert. Les théorèmes de Jordan-Hölder et Schreier. Revue Sci. 85, 579–596 (1947).

The first parts of the paper are largely an expository discussion of the theorems of Jordan-Hölder and Schreier for groups and lattices. Later the case of equivalence relations is considered; a concept of normality is introduced for relations and this is used to prove a general theorem of Jordan-Hölder for composition series. *O. Ore.*

Byrne, Lee. Boolean algebra in terms of inclusion. Amer. J. Math. 70, 139–143 (1948).

A system of four axioms for Boolean algebra is given where the only undefined terms are a class of elements and a binary relation (inclusion). However, the axioms are not stated directly in terms of the undefined relation, but rather in terms of three defined notions 'Nx,' 'Cyz,' and 'Pyzw' which we intuitively correspond to the sentential functions 'x is a null element,' 'y and z are complementary elements' and 'y is a greatest lower bound of z and w.' The sufficiency of the axiom system is established. *B. Jónsson.*

Dieudonné, Jean. Compléments à trois articles antérieurs. I. Bull. Soc. Math. France 74, 59–64 (1946).

The present note should be read in conjunction with the author's earlier paper [same Bull. 70, 46–75 (1942); these Rev. 6, 144]; the page references below are to this paper. The author points out that B. H. Arnold obtained independently [Ann. of Math. (2) 45, 24–49 (1944); these Rev. 5, 147] results which yield special cases of his own theorems on the structure and isomorphisms of simple rings with minimal left ideals. By generalizing Arnold's methods he gives shorter proofs of theorems 4 and 5 [pp. 56 and 69], and readily deduces the classical Skolem-Noether theorem on the automorphisms of simple algebras. He also notes that a remark about the isomorphisms of subrings of quasi-simple rings [bottom p. 70, top p. 71] is not valid without additional proof, and supplies this proof.
S. A. Jennings (Vancouver, B. C.).

Amitzur, Shimshon. On unique factorization in rings. Riveon Lematematika 2, 28–29 (1948). (Hebrew)

It is proved that if an integral domain has unique prime factorization, and if every proper residue class ring satisfies the descending chain condition, then it is a principal ideal domain. *I. S. Cohen* (Cambridge, Mass.).

Rédei, L., und Szele, T. Algebraischzahlentheoretische Betrachtungen über Ringe. I. Acta Math. 79, 291–320 (1947).

A generalisation to general rings of number-theoretical functions is studied. It is shown that the only rings where all such functions are polynomials in the ring are the finite fields. The question of how to associate another ring S with the given ring R so that the polynomials in S represent all functions f such that $f(x)\varepsilon R$ for $x\varepsilon R$ is studied. The ring S cannot always be chosen to contain R, e.g., when R coincides with the ring of residue classes modulo an integer m. A partial solution of the problem is nevertheless obtained for this ring by first introducing a ring S of which R is a homomorphic image and then proceeding to a ring T which contains S. If m is a power of a prime p the problem is completely solved: the rings in question are for S the ring of integers and for T the ring K_p of rational numbers the denominators of which are prime to p. This result is based on the example of the function $\psi(x)=((x^p-x)/p)^p$ which has the property $\psi(x)\equiv\psi(y)(p^e)$ if $x\equiv y(p^e)$, where $e\geqq 2$ and x, y are integers. It is a function but not a polynomial in the ring of residue classes mod p^e. The proof of the result for this ring is achieved by actually constructing the polynomials in K_p which generate all the functions in the ring. These functions are connected with functions studied by Dickson [Dickson, Einführung in die Zahlentheorie, Teubner, Leipzig and Berlin, 1931, pp. 20 and 25]. The case of a general m is reduced to the case of p^e, but is not completely solved so far. The results are inspired by the theorem of Nagell [Nyt Tidsskr. Mat. 29B, 53–62 (1918)] that all number-theoretical polynomials are of the form $\sum a_i\binom{x}{i}$. [This result was also obtained by Pólya, Rend. Circ. Mat. Palermo 40, 1–16 (1915).] Different versions of this theorem, in particular one by Kempner [Trans. Amer. Math. Soc. 22, 240–288 (1921)] are investigated.

A converse problem is also studied: to find all m such that the residue class ring mod m contains a function represented by a given polynomial with rational coefficients. In addition a generalisation of a result by Kempner [loc. cit.] is studied concerning integral polynomials which vanish mod m for all values. *O. Todd-Taussky.*

Johnson, R. E., and Kiokemeister, F. The endomorphisms of the total operator domain of an infinite module. Trans. Amer. Math. Soc. 62, 404–430 (1947).

Let P be a division ring and let Ξ be an infinite-dimensional vector space over P with a countable basis $\xi_1, \xi_2, \cdots$ such that every element ξ of Ξ has a unique representation $\xi=\sum\alpha_i\xi_i$, $\alpha_i\varepsilon$P, where only a finite number of α_i are different from zero. Let Δ be the set of all P-endomorphisms a of Δ, $\xi\rightarrow a\xi$. If a fixed basis ξ_i of Ξ is used, every a of Δ can be characterized by a matrix with coefficients in P such that only a finite number of rows contain coefficients different from zero. A sequence of elements $a_1, a_2, \cdots$ of Δ is said to be algebraically summable if to every ξ in Ξ there corresponds an integer $N(\xi)$ such that $a_k\xi=0$ for all $k>N(\xi)$. Then the sum $a_1+a_2+\cdots$ can be defined by $(a_1+a_2+\cdots)\xi=a_1\xi+a_2\xi+\cdots$, where the sum on the right contains only a finite number of terms different from zero. The principal topic of the paper is a discussion of the endomorphisms of the ring Δ. This is a generalization of the question of the endomorphisms of a complete matric algebra of finite degree. An endomorphism Θ of Δ, $a\rightarrow a^\Theta$, is said to be a meromorphism if it forms a one-to-one mapping of Δ on a subring Δ^Θ. An endomorphism Θ is said to be complete if it maps every algebraically summable sequence $a_1, a_2, \cdots$ of elements of Δ on an algebraically summable sequence, and if $(a_1+a_2+\cdots)^\Theta=a_1^\Theta+a_2^\Theta+\cdots$. If a weak topology is introduced in Δ, this is the condition that Θ is continuous. It is shown that every nonzero endomorphism Θ of Δ is a complete meromorphism. A formula is given for all endomorphisms of Δ. In the case of automorphisms Θ, this yields a special case of a result of N. Jacobson [Amer. J. Math. 67, 300–320 (1945); these Rev. 7, 2]. If a fixed basis ξ_i of Ξ is used, and if α is a given element of P, the mapping $\sum\gamma_i\xi_i\rightarrow\sum\gamma_i\alpha\xi_i$, γ_i in P, represents an endomorphism $\bar{\alpha}$ of Ξ. These $\bar{\alpha}$ form a ring $\overline{\text{P}}$ anti-isomorphic to P; the ring $\overline{\text{P}}$ depends on the choice of the ξ_i. A $\overline{\text{P}}$-endomorphism Θ of Δ is an endomorphism of Δ such that $(\bar{\alpha}a)^\Theta=\bar{\alpha}a^\Theta$, $(a\bar{\alpha})^\Theta=a^\Theta\bar{\alpha}$

for all a in Δ and all $\bar{a}$ in $\bar{P}$. These $\bar{P}$-endomorphisms can be studied further. Finally, it is shown that Δ has no non-zero anti-endomorphism. *R. Brauer* (Ann Arbor, Mich.).

Pickert, Günter. Bemerkungen zum Algebrenbegriff. Math. Ann. **120**, 158–164 (1948).

The author studies some of the troubles that arise in modules and algebras over noncommutative rings without unit. For example: if a left linear form module over R is to have a basis in the ordinary sense, then R must have a right unit. He distinguishes between algebras in the narrow sense (defined by the usual associativity conditions) and algebras in the wide sense (defined by a multiplication table), and gives various relations and conditions for existence. Unmentioned is the possibility of having the scalars operate differently on left and right [cf. R. Brauer, Bull. Amer. Math. Soc. **48**, 752–758 (1942); these Rev. **4**, 70]. *I. Kaplansky* (Princeton, N. J.).

Loonstra, F. La définition d'un système hypercomplexe. Nederl. Akad. Wetensch., Proc. **51**, 342 = Indagationes Math. **10**, 131 (1948).

The author points out that the usual definition of an algebra A over a ring R requires that R be commutative unless $A^2=0$. [For a more extensive study of this question, see Pickert's paper reviewed above.] *I. Kaplansky.*

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: conséquences de la loi de monodromie; résumé de la théorie locale. C. R. Acad. Sci. Paris **226**, 535–537 (1948).

A number of consequences of the law of monodromy proved by the author in a previous note [same C. R. **225**, 1113–1115 (1947); these Rev. **9**, 326] are announced. The notation is that used in the note cited; the results announced include the following. A finite algebraic extension K/k is normal if and only if, for all prime ideal factors $\mathfrak{P}$ in K of the discriminant $D_{K/k}$, $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ is normal and depends only on $\mathfrak{p}$, and for all prime ideals $\mathfrak{P}$ of K such that $N(P) \leqq |D_K|^{n1/2}$, $[K_{\mathfrak{P}}:k_{\mathfrak{p}}]$ depends only on $\mathfrak{p}$; the Galois group $G_{K/k}$ is generated by the inertial groups corresponding to the prime ideals $\mathfrak{P}$ in K which divide $D_{K/k}$ and by the groups of decomposition corresponding to the prime ideals $\mathfrak{P}$ in K such that $N(\mathfrak{P}) \leqq |D_K|^{\frac{1}{2}}$.

The balance of this note contains a résumé of the definitions and results of the author's local non-Abelian class field theory previously announced [same C. R. **219**, 433–435 (1944); **220**, 28–30, 761–763 (1945); **221**, 737–739 (1945); **222**, 37–40, 165–167, 581–583, 626–628, 984–986 (1946); these Rev. **7**, 364, 429, 510]. *D. C. Murdoch.*

Zelinsky, Daniel. Nonassociative valuations. Bull. Amer. Math. Soc. **54**, 175–183 (1948).

The author proves the following theorem. If an algebra of finite order has a unity quantity and has a valuation inducing a rank one valuation of the base field, then the value loop of the algebra is commutative, associative, and Archimedean-ordered. The proof is based upon a theorem concerning ordered loops which are shown to be Archimedean ordered Abelian groups. The hypotheses of the theorem are put in proper perspective by examples of noncommutative, nonassociative algebras of arbitrary finite order over a field which have valuations with nonassociative value loops and which are ordered algebras.

O. F. G. Schilling (Chicago, Ill.).

Schafer, R. D. The exceptional simple Jordan algebras. Amer. J. Math. **70**, 82–94 (1948).

Jordan algebras have been introduced by A. A. Albert [Ann. of Math. (2) **48**, 546–567 (1947); these Rev. **9**, 77] who proved that a simple Jordan algebra is either a Jordan algebra of linear transformations [cf. A. A. Albert, Trans. Amer. Math. Soc. **59**, 524–555 (1946); these Rev. **8**, 63] or is an "exceptional" simple Jordan algebra. If an exceptional Jordan algebra is considered as a central algebra over its center, there exists an extension field $\mathfrak{K}$ of finite degree of the center such that the corresponding scalar extension of the algebra is the Jordan algebra $\mathfrak{M}_3^s$ consisting of the three-rowed Hermitian matrices with coefficients in the Cayley algebra over $\mathfrak{K}$. The product of two elements a, b of this algebra is defined as $ab=\frac{1}{2}(a\cdot b+b\cdot a)$, where $a\cdot b$ denotes the matrix product. The topic of the present paper is a further study of these exceptional central simple Jordan algebras $\mathfrak{A}$ over a nonmodular field $\mathfrak{F}$. It is shown that $\mathfrak{A}$ is a reduced algebra, that is, that the primitive idempotents remain primitive under scalar extension. If $\mathfrak{M}_3$ denotes the complete matric algebra of degree 3 over $\mathfrak{F}$ and if $\mathfrak{C}$ is any Cayley-Dickson algebra over $\mathfrak{F}$, the direct product $\mathfrak{M}_3\times\mathfrak{C}$ is an involutorial central simple algebra of order 72 over $\mathfrak{F}$. The involution J here is induced by the involution $x\to\bar{x}$ in $\mathfrak{C}$ and the involution $m\to pm'm^{-1}$ in $\mathfrak{M}_3$, where p is any nonsingular diagonal matrix in $\mathfrak{M}_3$. It is shown that the exceptional algebra $\mathfrak{A}$ is isomorphic to the algebra of all J-symmetric elements of such a product $\mathfrak{M}_3\times\mathfrak{C}$, with multiplication defined in a manner analogous to that used in $\mathfrak{M}_3^s$. An analogous statement holds for reduced Jordan algebras having $\mathfrak{M}_3^s$ as split algebra in the case of an arbitrary groundfield of characteristic different from two. If $\mathfrak{F}$ is formally real, there exist inequivalent Jordan algebras over $\mathfrak{F}$ with $\mathfrak{M}_3^s$ as split algebra. *R. Brauer.*

THEORY OF GROUPS

Ramanathan, K. G. On the product on the elements in a finite Abelian group. J. Indian Math. Soc. (N.S.) **11**, 44–48 (1947).

A proof of the theorem that the product of the elements of a finite Abelian group is equal to its unique element of order 2 if it has one, and to the identity otherwise; with applications to number theory by taking the group to be the multiplicative group of prime residue classes modulo k, or one of its subgroups. Theorem and applications alike are due to G. A. Miller [Ann. of Math. (2) **4**, 188–190 (1903)]. *G. Higman* (Manchester).

Rado, R. A theorem on Abelian groups. J. London Math. Soc. **22** (1947), 219–226 (1948).

The new theorem includes as special cases both the following known theorems. Helly's theorem: let $A_1, \cdots, A_n$ be n convex sets of points in Euclidean k-space, where $n \geqq k+2$; suppose that any $k+1$ of the sets A_r have at least one point in common; then all n sets A_r have at least one point in common. Stieltjes' theorem: let $A_1, \cdots, A_n$ be n arithmetic progressions of integers, infinite in both directions, where $n \geqq 3$; suppose that any two of the progressions A_r have at least one member in common; then all n progressions A_r

have at least one member in common. Stieltjes' theorem is also generalized in another direction, to a statement about lattices (in the crystallographic sense).

H. S. M. Coxeter (Toronto, Ont.).

Piccard, Sophie. Les bases du groupe symétrique et du groupe alterné, dont l'une des substitutions est formée de deux transpositions. C. R. Acad. Sci. Paris **226**, 146–148 (1948).

If $T=(a_1a_2)(a_3a_4)$ and S is a product of one, two or three cycles the author gives necessary and sufficient conditions for S and T to generate the alternating and symmetric groups on n symbols. Certain exceptions occur for values of $n<9$. *G. de B. Robinson* (Toronto, Ont.).

Garnir, Henri. Une question de théorie des groupes et son application à un problème de vibrations posé par la chimie théorique. II. Bull. Soc. Roy. Sci. Liége **15**, 455–464 (1946).

In part I [same Bull. **15**, 357–382 (1946); these Rev. **8**, 562] representations of certain finite groups of two and three rowed matrices were analyzed. In the present paper there is a discussion of the way that such representations arise in the theory of vibrations with especial attention to molecular structure. A detailed treatment is given for the group of symmetries of a regular pyramid with a square base. *R. M. Thrall* (Ann Arbor, Mich.).

Frucht, Roberto. On the construction of partially ordered systems with a given group of automorphisms. Revista Unión Mat. Argentina **13**, 12–18 (1948). (Spanish)

G. Birkhoff [same Revista **11**, 155–157 (1946); these Rev. **7**, 411] showed that, given a group G consisting of α elements, there exists a partially ordered system P of $\alpha^2+\alpha$ elements whose group of automorphisms is isomorphic to G. The author shows that the number of elements in P can be reduced to α^2. *P. M. Whitman* (Silver Spring, Md.).

Faddeev, D. K. On the structure of groups of order p^nq. Doklady Akad. Nauk SSSR (N.S.) **58**, 533–534 (1947). (Russian)

The paper consists of a proof of the following theorem. If a group of order p^nq (p, q prime) contains no normal subgroup of order p^n nor one of order q, then $q\equiv 1 \pmod p$ and $pf/(p,f)\leqq n-1$, where f is the least positive integer such that $p^f\equiv 1 \pmod q$. Consequently, for fixed n, there exist only finitely many groups of order p^nq which possess no normal Sylow subgroup. The final paragraph suggests a generalization to arbitrary solvable groups.

R. A. Good (College Park, Md.).

Ado, I. D. Proof of the countability of a locally finite p-group with the minimal condition for normal divisors. Doklady Akad. Nauk SSSR (N.S.) **58**, 523–524 (1947). (Russian)

In a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 471–473 (1946); these Rev. **8**, 437] the author studied countable locally finite p-groups with the minimal condition on normal subgroups. He now shows that the hypothesis of countability was redundant. He first proves the following lemma: if an uncountable Abelian p-group A is a subgroup of finite index in a group G, then G does not have the minimal condition on normal subgroups. This is done by manipulating the subgroups $A_i\subset A$ consisting of elements of order p^i. To prove the theorem itself, he takes a minimal normal subgroup of finite index, and forms its upper central series $Z_1\subset Z_2\cdots$, using here one of his earlier results [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 299–301 (1943); these Rev. **6**, 146]. The possibility that Z_2/Z_1 is uncountable is ruled out by the use of the lemma. The remaining case is treated by theorems of Prüfer, Grün, and a theorem in the first paper cited above.

I. Kaplansky (Princeton, N. J.).

Dicman, A. P. On Sylow's theorem. Doklady Akad. Nauk SSSR (N.S.) **59**, 1235–1236 (1948). (Russian)

[The author's name has previously appeared as Dietzmann in non-Russian publications.] As a sequel to an earlier paper [Ann. of Math. (2) **48**, 137–146 (1947); these Rev. **8**, 436] the author seeks conditions, not only sufficient but also necessary, to insure the validity of a generalized form of Sylow's theorem. Typical is the first theorem, which, in the notation of the earlier paper, states: let $\{P_\beta\}$ be a class of conjugate maximal $|p; H_A;$ norm.$|$-subgroups of the group G; all the maximal $|p; H_A;$ norm.$|$-subgroups of G are conjugate if and only if, to every maximal $|p; H_A;$ norm.$|$-subgroup II of the group G, there corresponds a subgroup P_Π belonging to the class $\{P_\beta\}$ with the property that there exist only finitely many subgroups which are conjugate to P_Π in the group generated by P_Π and II. Other theorems give, with the imposition of suitable restrictions, analogous conditions necessary and sufficient that all the maximal $|p; H_A;$ inv.$|$-subgroups of G be conjugate.

R. A. Good (College Park, Md.).

Whitman, Philip M. Groups with a cyclic group as lattice-homomorph. Ann. of Math. (2) **49**, 347–351 (1948).

If G is a group, then let $L(G)$ be the partially ordered set of all the subgroups of G. The author defines as a lattice homomorphism of the group G upon the group H a single valued mapping f of $L(G)$ upon $L(H)$ which preserves intersection and composition of subgroups. [It should be noted that homomorphisms of the group G upon the group H will, in general, not induce lattice homomorphisms.] If G is finite, S a subgroup of H, then the cross cut Sf^{-1} of all the subgroups X of G such that $Xf=S$ is a well determined subgroup of G which is mapped by f upon S [a like remark holds for the compositum of all these X's]. The author's principal result: if G and H are finite, and if H is cyclic, then Hf^{-1} is cyclic and contains a subgroup lattice isomorphic to H. *R. Baer* (Urbana, Ill.).

Kontorovič, P. On the theory of noncommutative groups without torsion. Doklady Akad. Nauk SSSR (N.S.) **59**, 213–216 (1948). (Russian)

The author shews that considerable parts of the theory of completely reducible Abelian groups without torsion as developed by R. Baer [Duke Math. J. **3**, 68–122 (1937)] remain valid for a much wider class of groups which he calls R-groups. A group G without torsion (i.e., without elements of finite order, other than 1) is an R-group if it can be represented in the form of a set-theoretical sum of locally cyclic subgroups, co-prime in pairs (a "locally cyclic" group is an Abelian group of rank 1 or a rational group (Baer)). Examples of R-groups are: Abelian and metabelian groups without torsion, the infinite soluble S-groups investigated by the reviewer [Proc. London Math. Soc. (2) **44**, 53–60, 336–344 (1938); **49**, 184–194 (1946); these Rev. **8**, 132] and more generally groups which appear as union of a finite or infinite ascending central series all factor-groups of which are without torsion. Free groups and locally free

groups are R-groups; direct and free products of R-groups are again R-groups.

The author proves that a group G without torsion is an R-group if and only if the centraliser of each of its elements is an isolated subgroup. (The subgroup H is "isolated in G" if every equation $x^n=h$, where n is an integer and $h\neq 1$ an element of H, when solved in G has all its solutions in H.) In other words: if arbitrary powers of two elements are permutable, the elements themselves are permutable. Also: G is an R-group, if and only if the equation $x^n=a$, $a\in G$, if soluble at all, has a unique solution.

This theorem allows the classification of the elements and locally cyclic components of an R-group by means of Steinitz-numbers and "genera." [For a definition of, and operations on, these generalised numbers see Baer, loc. cit., 71–72]. The author then defines the terms "completely reducible" R-group (direct product of any numbers of factors such that in each factor the elements other than 1 have all the same genus) and "locally reducible" R-group (Baer: "separable"; every finite set of its elements belongs to a completely reducible direct factor) and introduces for a given genus the four fundamental element-complexes, and characteristic subgroups generated by them, as considered by Baer, but with an improved notation. He develops the theory of completely reducible R-groups on the lines of the Abelian case as far as the noncommutative character permits. If the maximal condition holds for the set of genera of G, further analogies are established. It is then possible to give necessary and sufficient conditions for R-groups to be completely reducible in terms of relations between the four fundamental subgroups. In particular, every direct factor of a completely reducible group G with maximal condition for its set of genera is itself completely reducible. No proofs are given. *K. A. Hirsch.*

Baer, Reinhold. The role of the center in the theory of direct decompositions. Bull. Amer. Math. Soc. **54**, 167–174 (1948).

In an earlier paper [Trans. Amer. Math. Soc. **62**, 62–98 (1947); these Rev. **9**, 134] the author proved the sufficiency of a certain condition for the existence of exchange isomorphic refinements to any two given direct decompositions of a loop L with operator domain M (briefly, an M-loop L). This condition, termed postulate E, is that, if an M-subloop S is a direct summand of L, then any pair of direct decompositions of S should have canonical refinements. The refinements $A=A'\oplus A''$, $B=B'\oplus B''$, $D=D'\oplus D''$, $E=E'\oplus E''$ constitute canonical refinements of the direct decompositions $L=A\oplus B=D\oplus E$ if the conditions

$$A'\oplus B'=B'\oplus D'=D'\oplus E'=E'\oplus A',$$
$$A''\oplus D''=D''\oplus E''=E''\oplus B''=B''\oplus A''$$

are satisfied. In the present note, the author sets up a one-to-one correspondence between the canonical refinements of a given pair of direct decompositions of an M-loop L and the canonical refinements of the pair of induced decompositions of the center $Z(L)$ of L. He deduces that a sufficient condition for Postulate E to hold for L is that it should hold for $Z(L)$. *D. Rees* (Manchester).

NUMBER THEORY

Beeger, N. G. W. H. Second extension of the table of least exponents ξ for which $2^\xi\equiv 1 \pmod p$. Nieuw Arch. Wiskunde (2) **22**, 310–311 (1948).

This table extends a previous one [same Arch. (2) **20**, 307–308 (1940); these Rev. **3**, 66] and extends over all primes p between 309672 and 320000. The factor $p=318781$ of $2^{314}+1$ is noted. *D. H. Lehmer* (Berkeley, Calif.).

Uhler, H. S. On Mersenne's number M_{257} and cognate data. Bull. Amer. Math. Soc. **54**, 378–380 (1948).

The author announces that $2^{257}-1$ is composite as predicted by Mersenne in 1644. The 226th term of the sequence 4, 14, 194, $\cdots$ is not divisible by $2^{257}-1$. The actual remainder of this division is given as well as the corresponding remainder for the composite number $2^{167}-1$. The tenth term of the above sequence is given in full together with the value of 2^{257}. [Cf. the following review.]

D. H. Lehmer (Berkeley, Calif.).

Uhler, H. S. On all of Mersenne's numbers particularly M_{199}. Proc. Nat. Acad. Sci. U. S. A. **34**, 102–103 (1948).

Just over 300 years ago Mersenne published his now famous statement that of the 55 numbers of the form 2^p-1, with p a prime not exceeding 257, only 11 are prime and they are given by $p=2$, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. At last with the present paper we have the complete story on the accuracy of this statement with the completion of the author's investigation, begun in 1944, of the six outstanding cases $p=157$, 167, 193, 199, 227 and 229. The test of Lucas is applied to $2^{199}-1$ to show that this number is composite. Thus Mersenne made just 5 mistakes in including $p=67$, 257 and excluding $p=61$, 89, 107. Twelve Mersenne numbers are primes and twelve are composite but without known factors at the present time.

D. H. Lehmer (Berkeley, Calif.).

Kaprekar, D. R. A peculiar gap-filling process for powers of (9)^n. J. Univ. Bombay (N.S.) **13**, part 5, Sect. A., 1 (1945).

Krishnaswami Ayyangar, A. A., and Kaprekar, D. R. Demlofication of Fibonacci numbers. J. Univ. Bombay (N.S.) **14**, part 5, Sect. A., 6–10 (1945).

Kaprekar, D. R. Some interesting applications of diagonalisation method. J. Univ. Bombay (N.S.) **15**, part 3, Sect. A., 7–9 (1946).

Notes on digital properties of numbers.

D. H. Lehmer (Berkeley, Calif.).

Tietze, Heinrich. Über vollständige Vielfachen-Mengen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 19–20 (1944).

A "vollständige Vielfachen-Menge" (in short, an $\mathfrak{M}$-set) is a set of positive integers $\mathfrak{M}=\{m_1, m_2, \cdots, m_\nu, \cdots\}$, such that for every pair of relatively prime integers a, b, both not zero, $\mathfrak{M}$ contains an integer m_ν which is divisible by a and is relatively prime to b. The author gives a simple proof of the following theorem. For every $\mathfrak{M}$-set $\mathfrak{M}_0$ and for every positive ϵ, there exists a subset $\mathfrak{M}_0(\epsilon)$ of $\mathfrak{M}_0$ which is a $\mathfrak{M}$-set and which has the property that the inequality $\phi(m)<\epsilon m$ holds for every integer m belonging to $\mathfrak{M}_0(\epsilon)$. Here $\phi(m)$ denotes the Euler function.

A. L. Whiteman (Los Angeles, Calif.).

Petr, K. On the Bernoulli polynomials. Rozpravy II. Třídy České Akad. **53**, no. 40, 16 pp. (1943). (Czech)

A German translation appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 511–526 (1943); these Rev. **8**, 441.

Bambah, R. P. On integer cube-roots of the unit matrix. Math. Student **14** (1946), 69–70 (1948).

The cube of a matrix A of order 2 with integral elements is the identity matrix I if and only if $|A|=1$ and the trace of A is -1, unless $A=I$. *I. Niven* (Eugene, Ore.).

Bambah, R. P., and Chowla, S. Some new congruence properties of Ramanujan's function $\tau(n)$. Math. Student **14**, 24–26 (1946).

Congruence properties of the usual type, connecting Ramanujan's $\tau(n)$ with the sum $\sigma_k(n)$ of the kth powers of the divisors of n, are given with respect to the moduli 2^5, 3^4, 5^3 and 7^3. All these results except the last follow from the tables of D. B. Lahiri [Bull. Calcutta Math. Soc. **39**, 33–52 (1947); these Rev. **9**, 225]. This is true even of the conjecture $\tau(n)\equiv\sigma_{11}(n) \pmod{2^8}$ (n odd) made by the authors. For the modulus 49 they give

$$\tau(n)\equiv 8n^4\sigma_3(n)-14\{2(1-n-n^2)\sigma_3(n)+(2n^2-3)\sigma(n)+\Sigma\} \pmod{49},$$

where n is not divisible by 7 and $\Sigma=\sum_{u+7v=n}\sigma(u)\sigma_3(7v)$. No proofs are given. *D. H. Lehmer* (Berkeley, Calif.).

Jarden, Dov, and Katz, Alexander. New factors of Fibonacci numbers. Riveon Lematematika **2**, 35 (1948). (Hebrew)

The factors 1069, 29717 and 27941 are announced for u_{89}, u_{117} and u_{127}, respectively. No other new factors of Fibonacci's u_n and $v_n=u_{2n}/u_n$ exist below 10^5. [See the same Riveon 1, 35–37 (1946); 2, 22 (1947); these Rev. 8, 135; 9, 134.] *D. H. Lehmer* (Berkeley, Calif.).

Williams, G. T., and Browne, D. H. A family of integers and a theorem on circles. Amer. Math. Monthly **54**, 534–536 (1947).

"Between two tangent unit circles lying on a line L, a third circle is inscribed tangent to both circles and to L. Then two more circles are inscribed, each tangent to two circles and to L. Now four circles are inscribed similarly along L, and so on." The sum of areas of the infinite set of circles is established as $A=\pi+\pi\zeta(3)/\zeta(4)$, $\zeta(s)=\sum n^{-s}$. The figure leads to sets of integers A_ν^n which are of some interest in themselves: A_ν^n is the νth entry in the nth row of the arrangement which will explain itself:

1				2				1
1		3		2		3		1
1	4	3	5	2	5	3	4	1
		...				...		.

The numbers are related to Farey series and Fibonacci numbers. Some of their properties are: S_n = sum of nth row $=3^n+1$; $A_{r-1}^n+A_{r+1}^n\equiv 0 \pmod{A_r^n}$; if l, m are relatively prime positive integers, l, m occur in some row as neighboring elements; every integer $a>1$ appears from some n on exactly $\varphi(a)$ times in every row. *A. J. Kempner.*

Caris, P. A. Rational solutions of a Diophantine equation. Amer. Math. Monthly **55**, 238–240 (1948).

This paper is concerned with the Diophantine equation

$$u^3+v^3+w^3-3uvw=x^3+y^3.$$

The author derives in a simple manner the following general nontrivial rational solution: $u=acq-3bcq-cp^2-dp^2+adq+bdq$, $v=acq-3bcq-cp^2+dp^2+adq+bdq$, $w=acq-3bcq+2cp^2+adq+bdq$, $x=3acp-3bcp+adp+3bdp$, $y=2adp-6bcp$, where $pq=a^2+3b^2$. *A. L. Whiteman.*

Guttman, Solomon. Universal magic squares and multigrade equations. Scripta Math. **13**, 187–202 (1947).

A natural magic square is defined as a square containing in its cells the integers $1, 2, 3, \cdots$ in their natural order. Most of its properties do not belong to magic squares in the ordinary sense. A pattern is defined as a figure containing any number of cells marked $-$ and an equal number of cells marked $+$. Many illustrations of such patterns are given. They are divided in groups according to their ranks and rules are given for the determination of the rank. When a pattern of rank k is applied to a natural square and the integers in the $-$ cells are denoted by a_i, those in the $+$ cells by b_i, then $\sum a_i^m=\sum b_i^m$ for $m=1, 2, \cdots, k$. Hence patterns yield solutions of the Tarry-Escott problem (multigrade equalities). When a pattern is moved from one position to another within the same square each a_i and b_i is increased by the same number h and $\sum_i(a_i+h)^m=\sum_i(b_i+h)^m$ for $m=1, 2, \cdots, k$. Natural squares possess a "magic constant" $c=\sum a_i^{k+1}-\sum b_i^{k+1}$. The numbers (a, b) may represent $ax+by$; hence patterns yield general solutions of the Tarry-Escott problem. *N. G. W. H. Beeger.*

Palamà, G. Teoremi relativi alle uguaglianze multigrade. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 366–394 (1947).

Proofs of known elementary theorems on multi-degree equalities (Tarry-Escott problem) and some new theorems inferred from them. Numerical examples.

N. G. W. H. Beeger (Amsterdam).

***Gloden, A., et Palamà, G. Bibliographie des Multigrades avec Quelques Notices Biographiques.** A. Gloden, Luxembourg, 1948. iv+64 pp. 50 Belgian francs.

Châtelet, François. Utilisation des congruences en analyse indéterminée. Ann. Univ. Lyon. Sect. A. (3) **10**, 5–22 (1947).

Let C be a rational cubic of genus one with group G of rational points and, C reducing to a cubic Γ over the finite field k of integers modulo p, let Γ have no double points. Using a modification of a method of E. Lutz [J. Reine Angew. Math. **177**, 238–247 (1937)] the author proves that the points of G whose homogeneous coordinates are congruent modulo p to those of the identity element of G (point at infinity) form a subgroup G_p, and that the quotient group G/G_p is isomorphic to a subgroup of G_h, the group of points of the cubic Γ in k. A new proof is given of the known result that the elements of G of finite order (the exceptional points) have integral nonhomogeneous coordinates. Three numerical examples are discussed. *I. Niven* (Eugene, Ore.).

Châtelet, François. Intérêt et signification de l'analyse indéterminée. Rev. Gén. Sci. Pures Appl. N. S. **54**, 199–201 (1947).

Inkeri, K. Untersuchungen über die Fermatsche Vermutung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **33**, 60 pp. (1946).

This memoir is concerned with a large number of rather unconnected results on the Fermat problem, dealing mostly with conditions on the integers x, y, z in (1) $x^l+y^l+z^l=0$

(l an odd prime) and (2) $x^m+y^m+z^m=0$ ($m=l^n$). The main results are as follows. (a) If x is not divisible by l but is divisible by the prime p then $p^{l-1}-1$ is divisible by l^2, in case (2) holds. This is a generalization of a theorem of Furtwängler ($m=1$). Three other generalizations of the same sort are given. (b) If $l\not\equiv 1, 9 \pmod{20}$, one of x, y, z in (1) is divisible by 5. (c) If $(n\log l)^2>l$ then (2) has no solution. Other results involve inequalities. It is shown that x, y, z in (1) all exceed $(30l^2+1)^l$ and in case one of x, y, z is divisible by l they all exceed $\frac{1}{3}l^{2l-4}$. In particular, x, y, z each have more than 5000 digits and in case they are prime to l they have more than 4.6×10^9 digits. There is a bibliography of 38 titles, mostly recent. *D. H. Lehmer.*

Racliş, Nicolas. Remarques sur le premier théorème de Fermat. Acad. Roum. Bull. Sect. Sci. **26**, 281–282 (1946).

Racliş, Nicolas. Méthode du criblage. Acad. Roum. Bull. Sect. Sci. **26**, 360–362 (1946).

Announcements of results discussed in more detail in Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] **14**, 3–9, 145–156 (1943); **15**, 3–21 (1944); these Rev. **7**, 47.

Vandiver, H. S. Applications of cyclotomy to the theory of nonhomogeneous equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. **34**, 62–66 (1948).

Let p be an odd prime belonging to the exponent $t \pmod l$ and such that $p^t=1+cl$, where l is odd and prime to c. Let g be a primitive root in a finite field $F(p^t)$ and denote by $[v, r]$ the number of distinct solutions g^α, g^γ of $1+g^{\alpha+u}=g^{\gamma+lv}$, α being in the set $0, 1, \cdots, l-1$ and γ in the set $0, 1, \cdots, c-1$. Finally set $B_{h,k}=\sum_{i,j}[i+h, j+k][i, j]$. It is then proved by the author that $B_{h,k}=cl-1$ for $h\not\equiv 0 \pmod c$ and $k\not\equiv 0 \pmod l$; similar results are derived if $h\equiv 0 \pmod c$, or $k\equiv 0 \pmod l$, or both. These quadratic relations between the solution numbers $[i, j]$ were referred to in an earlier paper by the author [same Proc. **32**, 317–319 (1946); these Rev. **8**, 313] and their proof is based on results established in that paper. *H. W. Brinkmann.*

Vandiver, H. S. New types of congruences involving Bernoulli numbers and Fermat's quotient. Proc. Nat. Acad. Sci. U. S. A. **34**, 103–110 (1948).

Let the Bernoulli numbers b_k be defined symbolically by $(b+1)^n=b^n$, where $b^k=b_k$. Various congruences involving these numbers are derived in this paper, mainly by the use of a relation derived by the author in an earlier paper [Duke Math. J. **8**, 575–584 (1941); these Rev. **3**, 67]. A typical result is $(mb+k)^n\equiv\sum_{a=1}^{p-2}(ma+k)^nR_a \pmod p$; here p is an odd prime, $R_a=1+\frac{1}{2}+\cdots+1/a$ and the left-hand side is to be interpreted symbolically. A very simple result is $b_0+b_1+\cdots+b_{p-2}\equiv -1 \pmod p$, which is apparently new. Congruences are given that involve the Fermat quotient $q(m)=(m^{p-1}-1)/p$, $(m, p)=1$, and the Wilson quotient $W=((p-1)!+1)/p$. A sample of these is $-q(m)+W+\frac{1}{2}-\sum_{a=2}^{p-2}m^ab_a/n\equiv\sum_\rho\rho/(\rho^p-1) \pmod p$, where the sum on the right is extended over the distinct mth roots ρ of unity, excepting unity. *H. W. Brinkmann* (Swarthmore, Pa.).

Bell, E. T. The problem of Liouville's theorems on arithmetical quadratic forms. Scripta Math. **13**, 177–185 (1947).

Bell, E. T. Analogues for arithmetical functions of the elementary transcendental functions. Math. Mag. **21**, 65–73 (1947).

For applications in combinatorial analysis to be given in a later paper, the author recapitulates his algebra of numerical functions [cf. J. Indian Math. Soc. (1) **17**, 249–260 (1928) for further references] and defines analogues of the elementary transcendental functions in that algebra.

P. Hartman (Baltimore, Md.).

Basoco, M. A. On certain arithmetical functions due to M. Georges Humbert. J. Math. Pures Appl. (9) **26** (1947), 237–250 (1948).

Humbert's functions are entire functions which are related to the theta functions by the functional equations which determine them. The author displays these functions along with their functional equations. He also obtains identities involving these functions and theta functions from which he is able to find formulas connected with the representation of an integer as a sum of five squares.

H. S. Zuckerman (Seattle, Wash.).

Dyson, F. J. The approximation to algebraic numbers by rationals. Acta Math. **79**, 225–240 (1947).

C. L. Siegel [Math. Z. **10**, 173–213 (1921)] zeigte: Ist ξ algebraisch n-ten Grades, so hat die Ungleichung $|\xi-p/q|<q^{-\mu}$ für $\mu\geqq 2n^1$ höchstens endlich viele Lösungen in ganzen rationalen Zahlen p, q. Dyson verschärft nun den Exponenten auf $\mu>(2n)^{\frac{1}{2}}$. Der Beweis stützt sich auf einen Hilfssatz über Polynome in 2 Veränderlichen.

T. Schneider (Göttingen).

Hinčin, A. Ya. On the theory of linear Diophantine approximations. Doklady Akad. Nauk SSSR (N.S.) **59**, 865–867 (1948). (Russian)

Let θ_{ij} and α_j ($1\leqq i\leqq m, 1\leqq j\leqq n$) be real numbers, and put

$$S_j=\sum_{i=1}^{m}\theta_{ij}x_i-y_j, \quad T_i=\sum_{j=1}^{n}\theta_{ij}u_j-v_i,$$

$$u=\max_{1\leqq j\leqq n}|u_j|, \quad T=T(u_j, v_i)=\max_{1\leqq i\leqq m}|T_i|,$$

$$A=A(u_j)=|\sum_{j=1}^{n}\alpha_ju_j-w|,$$

where w is the integer nearest to $\sum_{j=1}^{n}\alpha_ju_j$, hence $0\leqq A\leqq\frac{1}{2}$. Denote by $\varphi(t)$ a positive monotonic increasing function of the positive variable t, and by $\psi(t)$ the function inverse to $t\varphi(t)$. Then the following theorem is proved. There exist positive constants c_1 and c_2 such that the system of inequalities

$$|S_j-\alpha_j|<c_1/t, \quad |x_i|<c_2\varphi(t), \qquad 1\leqq i\leqq m, 1\leqq j\leqq n,$$

is solvable for every $t>0$ in integers x_i, y_j if and only if there exists a positive constant Γ such that, for every system of integers u_j, v_i,

$$A\begin{cases}=0, & uT=0,\\ <\Gamma u/\psi(u/T), & uT>0.\end{cases}$$

K. Mahler (Manchester).

Davenport, H. Non-homogeneous binary quadratic forms. IV. Nederl. Akad. Wetensch., Proc. **50**, 741–749, 909–917 = Indagationes Math. **9**, 351–359, 420–428 (1947).

[For papers II and III of this series see the same Proc. **50**, 378–389, 484–491 = Indagationes Math. **9**, 236–247, 290–297 (1947); these Rev. **8**, 565; **9**, 79. For definitions, formulae and notation see the review of paper II, of which this

paper is a continuation.] The paper deals with the minima $M(a, b)$ of $|(\xi-a)(\xi'-b)|$. Suppose that a, b are not of the form (2), nor of the form (4) $a=\tau/a_m+\xi_0$, $b=\tau'/a_m'+\xi_0'$ (or vice versa), where τ is a unit and ξ_0 an integer of $k(\theta)$, and where m is an odd positive integer and a_m is defined by $a_m=2(\theta^{n+1}-1)/(\theta^n+1)$ $(n=3m)$; then the lower bound $M(a, b)$ (for integers ξ of $k(\theta)$) satisfies $M(a, b)\leqq\frac{1}{4}\theta^{-1}$. If a, b are of the form (2), we have $M(a, b)=\frac{1}{4}$; if a, b are of the form (4), we have $M(a, b)=1/|a_m a_m'|$ and in both cases this minimum is attained for an infinity of integers ξ of $k(\theta)$. This theorem not only solves the question of the third minimum which was left open in paper II, but establishes the existence of an infinite sequence of minima which may be expressed in terms of the Fibonacci numbers, as the author points out. *J. F. Koksma* (Amsterdam).

Davenport, H., and Heilbronn, H. Asymmetric inequalities for nonhomogeneous linear forms. J. London Math. Soc. **22**, 53–61 (1947).

Let $\alpha, \beta, \gamma, \delta$ be real numbers with $\Delta=\alpha\delta-\gamma\beta\neq 0$ and let ξ, η denote the linear forms $\xi=\alpha x+\beta y$, $\eta=\gamma x+\delta y$. Minkowski proved that for any real numbers λ, μ there exist integers x, y for which

$$(1) \qquad |(\xi+\lambda)(\eta+\mu)|\leqq\tfrac{1}{4}|\Delta|$$

and it is plain that this result is the best possible, as is shown by the case $\xi=x$, $\eta=y$, $\lambda=\mu=\frac{1}{2}$. Now the authors prove that if one replaces (1) by the inequality

$$(2) \qquad (\xi+\lambda)(\eta+\mu)<|\Delta|,$$

there always exist integers x, y for which not only (2) holds, but also the supplementary conditions $\xi+\lambda\geqq 0$, $\eta+\mu\geqq 0$. Moreover, they prove that for any $c<1$ there exist $\alpha, \beta, \gamma, \delta, \lambda, \mu$ such that the inequalities

$$(\xi+\lambda)(\eta+\mu)<c|\Delta|, \quad (\xi+\lambda)(\eta+\mu)\geqq 0$$

cannot hold simultaneously for integers x, y.

J. F. Koksma (Amsterdam).

Chalk, J. H. H. On the positive values of linear forms. Quart. J. Math., Oxford Ser. **18**, 215–227 (1947).

Let $L_1, \cdots, L_n$ be n homogeneous linear forms in n variables $u_1, \cdots, u_n$ with determinant $\Delta\neq 0$. Then, if $c_1, \cdots, c_n$ are any real constants, there exist integral values of $u_1, \cdots, u_n$, such that simultaneously $L_1+c_1>0, \cdots, L_n+c_n>0$, $(L_1+c_1)(L_2+c_2)\cdots(L_n+c_n)\leqq|\Delta|$. (Putting $c_1=\cdots=c_n=0$ one sees that this theorem on inhomogeneous forms includes a nontrivial theorem on homogeneous forms.) The author further proves: unless the ratios of the coefficients in each of the forms $L_1, \cdots, L_n$ are all rational the result of the above theorem is valid with strict inequality. For $n=2$ these results were proved by B. Segre and by K. Mahler [Duke Math. J. **12**, 337–365, 367–371 (1945); these Rev. **6**, 258] in the homogeneous case and by Davenport and Heilbronn in the inhomogeneous case [cf. the preceding review]. *J. F. Koksma* (Amsterdam).

Skolem, Th. A proof of the algebraic independence of certain values of the exponential function. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 12, 40–43 (1947).

In this note the author gives a simple and short proof of the following result, equivalent to the well-known theorem of F. Lindemann. The numbers $e^{\alpha_1}, e^{\alpha_2}, \cdots, e^{\alpha_n}$ are algebraically independent if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are algebraic numbers, linearly independent relative to the field of rational numbers. The author's proof is a natural extension of the ordinary proof for the transcendence of e and π, well-known from textbooks, where a suitably chosen prime number plays an important part. *J. Popken* (Utrecht).

Rodosskiĭ, K. A. On the complex zeros of Dirichlet's L-functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 47–56 (1948). (Russian)

This paper is devoted to a generalization of a theorem of Linnik [Rec. Math. (Mat. Sbornik) N.S. **15(57)**, 3–12 (1944); these Rev. **6**, 260] on the number of the Dirichlet $L(s, \chi)$ functions having zeros near the line $R(s)=1$. The author's theorem is as follows. Let λ be an arbitrary number between 0 and 1, $\psi(D)$ a number satisfying the inequalities $\frac{1}{3}\log D\leqq\psi(D)\leqq(\log D)^\lambda$, and T_1 a real number such that $|T_1|=T\geqq\log^4 D$; then the number of L-functions (mod D) which have zeros in the rectangle $1-(\psi(D)/\log DT)\leqq\sigma\leqq 1$, $|t-T_1|\leqq\log^4 DT$, does not exceed $\exp\{A(\lambda)\psi(D)+28\log\log DT\}$, where $A(\lambda)$ is a constant depending only on λ. This differs from Linnik's theorem in the presence of the parameter T_1. The proof follows very closely the method used by Linnik in the paper referred to. Each of Linnik's eight lemmas is extended to apply to the broader situation. The present paper is actually somewhat less difficult to read than Linnik's, since more of the details of proof are given.

The result stated has the following consequence, important for the study of primes in arithmetic progressions. If ϵ is a fixed number between 0 and $\frac{1}{3}$ and T_1 is any real number such that $|T_1|\leqq D^{\epsilon^{-1}-1}$, then there is an absolute constant A such that for sufficiently large D at most $D^{A\epsilon}$ of the L-functions (mod D) have zeros in the rectangle $1\geqq\sigma\geqq 1-\epsilon$, $|t-T_1|\leqq\log^4 D$. Linnik's theorem itself gives only the weaker form of this result in which the condition $|T_1|\leqq D^{\epsilon^{-1}-1}$ is replaced by the condition $T_1=O(\log^c D)$ for some c. *P. T. Bateman* (Princeton, N. J.).

Banerjee, D. P. On the divisors of numbers. Bull. Calcutta Math. Soc. **39**, 57–58 (1947).

Let $E(n)$ be the excess of the number of divisors of n of the form $4m+1$ over the number of divisors of the form $4m+3$. Clearly, $E(n)=\sum_{d|n}\sin\frac{1}{2}d\pi$ and

$$\sum_1^\infty E(n)q^n=\sum_1^\infty q^m(1-q^m)^{-1}\sin\tfrac{1}{2}m\pi.$$

Starting from this formula and using some Jacobian identities, the author derives other identities involving $\sum_{d|n}d\cos\frac{1}{2}d\pi$, $\sum_{d|n}d\sin\frac{1}{2}d\pi$. *R. Bellman.*

Tietze, Heinrich. Eine Bemerkung zum Lehmerschen Problem. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 163–170 (1944).

The Lehmer problem consists of the determination of the asymptotic order as $x\to\infty$ of the sum $S_1(x)=\sum A_1(n)$, $n\leqq x$, $n\varepsilon G$, where $A_1(n)$ is the number of squarefree divisors of n, and G is some subset of the positive integers. The problem was solved in some particular cases by Lehmer [Amer. J. Math. **22**, 293–335 (1900)] and in the general case by Landau [Amer. J. Math. **31**, 86–102 (1909)]. The author considers the general sum $S_g(x)=\sum A_{g-1}(n)$, $n\leqq x$, $n\varepsilon G$, where $A_g(n)$ is the number of gth-power free divisors of n. Analytic methods are used. *R. Bellman.*

Ren'i, A. On the representation of an even number as the sum of a single prime and a single almost-prime number. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 57–78 (1948). (Russian)

[The author's name appears in non-Russian publications as Rényi.] In this paper are the details of the proofs sketched

in an earlier paper of the same title by the author [Doklady Akad. Nauk SSSR (N.S.) 56, 455–458 (1947); these Rev. 9, 136]. In addition to the previously announced result that every even number is the sum of a prime and an almost-prime, the author states that his method yields the results that every odd number is the sum of a prime and twice an almost-prime and that for each fixed integer m (positive or negative) there exists an infinity of primes p such that $p+m$ is almost prime.

In addition to Brun's sieve and the results of Titchmarsh, Page and Siegel on the Dirichlet L-series, the author uses a generalization of the "great sieve" due to U. V. Linnik [C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 292–294 (1941); Rec. Math. [Mat. Sbornik] N.S. 15(57), 3–12 (1944); these Rev. 2, 349; 6, 260] to enable him to prove a certain result about the L-series. [See the earlier review.]

L. Schoenfeld (Urbana, Ill.).

Bellman, Richard, and Shapiro, Harold N. On a problem in additive number theory. Ann. of Math. (2) 49, 333–340 (1948).

The authors consider the arithmetical function $\alpha(n)$, defined as the number of terms in the binary representation of n, which is in some respects analogous to $d(n)$, the number of divisors of n. They write $\alpha_r(n)=\alpha(\alpha_{r-1}(n))$, $\alpha_0(n)=n$, $A_r(x)=\sum_{n\leq x}\alpha_r(n)$, and obtain the following results for $x\to\infty$: (1) $A_1(x)=x\log x/2\log 2+O(x\log\log x)$; (2) $\sum_{n\leq x}2^{-n}A_2(2^n)\sim x\log x/2\log 2$; (3) $A_r(x)$ is not asymptotic to any elementary function for $r\geqq 2$. Result (1) is established by making use of the fact that $\alpha(2^L+r)=1+\alpha(r)$ if $1\leqq r<2^L$. It is stated in a footnote that the error term can, in fact, be reduced to $O(x)$. Result (2) is proved by means of a Tauberian theorem due to Hardy and Littlewood [Proc. London Math. Soc. (2) 13, 174–191 (1914), theorem 8]. Result (3) (which, as the authors observe, is surprising since iteration might be expected to have a smoothing effect) is proved in several stages. It is first shown that, if $A_r(x)$ is asymptotic to an elementary function, that function must be $x\log_r x/2\log 2$, where $\log_r x=\log(\log_{r-1}x)$, $\log_1 x=\log x$; next it is shown that $A_2(x)$ is not asymptotic to any elementary function; finally (3) is deduced from the fact that the relation $A_r(x)\sim x\log_r x/2\log 2$ would imply $A_{r-1}(x)\sim x\log_{r-1}x/2\log 2$.

There is a misprint in the identity (3.1) which should read

$$\left(\sum_{k=0}^{\infty}\frac{x^{2^k}}{1+ax^{2^k}}\right)\prod_{k=0}^{\infty}(1+ax^{2^k})=\sum_{n=1}^{\infty}\alpha(n)a^{\alpha(n)-1}x^n.$$

Furthermore, the factor 2^{-n-1} has been omitted from the right-hand side of (6.1). This omission does not, however, affect the argument. *L. Mirsky* (Sheffield).

Radhakrishna Rao, C. Finite geometries and certain derived results in theory of numbers. Proc. Nat. Inst. Sci. India 11, 136–149 (1945).

Let $m=p^n$ (p a prime) and either $v=(m^{t+1}-1)/(m-1)$, $\theta=(m^{t+1}-1)/(m^2-1)$ and integral, $k=(m^{d+1}-1)/(m-1)$ or $v=m^t-1$, $\theta=(m^t-1)/(m-1)$, $k=m^d$. The author constructs sets $d_{11},\cdots,d_{1k};\cdots;d_{\theta 1},\cdots,d_{\theta k}$, such that the differences $d_{ij}-d_{il}$ contain all nonzero residues mod v which are not divisible by θ exactly λ_1 times and which are divisible by θ exactly λ_2 times. The numbers η, λ_1, λ_2 depend on v, θ, k and m. If θ is not integral then sets can be constructed such that every nonzero residue mod v occurs λ times. The methods are generalisations of those of J. Singer [Trans. Amer. Math. Soc. 43, 377–385 (1938)] and of R. C. Bose. The main idea is a representation of $PG(t,m)$ by the nonzero elements of $GF(m^{t+1})$, where x and ax represent the same point if a is in $GF(m)$ and of $EG(t,m)$ by the elements of $GF(m^t)$. These representations are easily obtained by considering $GF(m^{t+1})$ or $GF(m^t)$ as an algebraic extension of $GF(m)$. The author's configurations have important applications in the design of experiments. *H. B. Mann.*

Radhakrishna Rao, C. Difference sets and combinatorial arrangements derivable from finite geometries. Proc. Nat. Inst. Sci. India 12, 123–135 (1946).

[For the definition of difference sets cf. the preceding review.] In the present paper the author proves that not every difference set can be obtained from a finite geometry by his method and shows that for the case that $m=p^n$ is the power of a prime p, the derivation of difference sets depends on the solution of certain recurrence relations derived from polynomials irreducible mod p. He then shows that several of the known series of incomplete balanced block designs can be derived from difference sets. He finally discusses hypercubes of strength d, configurations useful in constructing factorial designs where interactions of order less than d remain unconfounded. *H. B. Mann* (Columbus, Ohio).

Min, Szu-Hoa. On the Euclidean algorithm in real quadratic fields. J. London Math. Soc. 22, 88–90 (1947).

The problem of determining all real quadratic Euclidean fields $P(\sqrt{m})$ is still unsolved only in the case where m is a prime of the form $8n+1$. Erdös and Ko [same J. 13, 3–8 (1938)] proved that only a finite number of such Euclidean fields exist. Hua [Trans. Amer. Math. Soc. 56, 537–546 (1944); these Rev. 6, 170] showed that the Euclidean algorithm does not exist if $p>e^{250}$. He stated that he was able to obtain the better bound e^{180}. In this paper the author outlines a proof that the algorithm does not exist if $p>e^{34}$ and the least quadratic nonresidue mod p is greater than 20.

A. Brauer (Chapel Hill, N. C.).

Carlitz, L., and Cohen, Eckford. The number of representations of a polynomial in certain special quadratic forms. Duke Math. J. 15, 219–228 (1948).

The forms have some coefficients in $GF(p^n)$ and others of degree 1 in $GF[p^n,x]$. The number of representations is determined by the singular series method developed by the first author [same J. 14, 1105–1120 (1947); these Rev. 9, 337]. *R. Hull* (Lafayette, Ind.).

ANALYSIS

***Lichnerowicz, A. Algèbre et Analyse Linéaires.** Masson et Cie., Paris, 1947. 316 pp. 800 francs.

While intended for physicists, this book is an extremely valuable introduction to linear mathematics for students of pure mathematics. The treatment is systematic and rigorous. It is divided into two parts, devoted to algebra and analysis, respectively. The first part treats linear equations, determinants and matrices, Hermitian forms, characteristic roots and resolvents, and tensor algebra. The second surveys the theory of exterior differential forms, the general form of Stokes's theorem and its specialization to two and three dimensions, orthogonal series, Fourier integrals,

bounded linear operators in Hilbert space, and the classical theory of integral equations for L^2 integrable kernels.
H. Pollard (Ithaca, N. Y.).

Baganas, Nicolas. Sur une identité d'Abel. C. R. Acad. Sci. Paris **226**, 1064–1066 (1948).

Let $R(z)$ be a polynomial whose zeros are all simple. The problem is that of determining all polynomials $\rho(z)$ such that the integral of $\rho/\sqrt{R}$ is a sum of logarithms of algebroid functions of finite order with a finite number of branch points. It is announced that the problem is equivalent to that of determining all pairs $f_1(z)$, $f_2(z)$ of integral functions of finite order such that $f_1^2-f_2^2R=1$. Various detailed results concerning f_1 and f_2 are stated. Other statements deal with Abel's integrability problem, in which the periodicity of the expansion of $\sqrt{R}$ as a continued fraction is the deciding factor. *J. F. Ritt* (New York, N. Y.).

Levin, V. I. Exact constants in inequalities of the Carlson type. Doklady Akad. Nauk SSSR (N.S.) **59**, 635–638 (1948). (Russian)

Let $f(x)\geqq 0$. The inequalities in question have the general form

$$(*)\qquad \int_0^\infty f(x)dx \leqq C\left\{\int_0^\infty x^{p-1-\lambda}f^p(x)dx\right\}^s\left\{\int_0^\infty x^{q-1+\mu}f^q(x)dx\right\}^t,$$

with $s=\mu/(p\mu+q\lambda)$, $t=\lambda/(p\mu+q\lambda)$; Carlson's inequality is the case $p=q=2$, $\lambda=\mu=1$. Special cases have been considered by Gabriel [J. London Math. Soc. **12**, 130–132 (1937)] and B. Nagy [Mat. Fiz. Lapok **48**, 162–175 (1941); these Rev. **7**, 8]; the existence of the general inequality (*) was proved by Caton [Duke Math. J. **6**, 442–461 (1940); these Rev. **2**, 75] and again by Bellman [Duke Math. J. **10**, 547–550 (1943); these Rev. **5**, 1], but the exact constant was known only in certain cases. Here the author shows that the exact value of C for $p>1$, $q>1$, $\lambda>0$, $\mu>0$ is, with $1-s-t=\alpha$,

$$(ps)^{-s}(qt)^{-t}\left\{\frac{\Gamma(s/\alpha)\Gamma(t/\alpha)}{(\lambda+\mu)\Gamma((s+t)/\alpha)}\right\}^\alpha.$$

If $p\to 1$ and $q\to 1$, $C\to 1$, and the inequality holds also for $p=q=1$.

The proof is, in outline, as follows. Let $p'=p/(p-1)$, $q'=q/(q-1)$, $\delta=p\mu+q\lambda-\mu-\lambda$. The function

$$z(k)=\{(1-k)^{p-1}/k^{q-1}\}^{1/\delta}$$

is strictly monotonic in $0<k<1$ and so has an inverse $k=k(z)$, $0<z<\infty$. For $y>0$ write

$$\int f(x)dx=\int k(yx)x^{-(p-1-\lambda)/p}x^{(p-1-\lambda)/p}f(x)dx + \int[1-k(yx)]x^{-(q-1+\mu)/q}x^{(q-1+\mu)/q}f(x)dx$$

and apply Hölder's inequality, with index p in the first integral on the right and index q in the second; we get $\int f(x)dx\leqq y^{-\lambda/p}I_1P+y^{\mu/q}I_2Q$, where P, Q are the factors involving the integrals on the right of (*) and I_1 and I_2 are the other factors, with y eliminated by the natural change of variable. If we take $y=(qtI_1P/psI_2Q)^{pq/(p\mu+q\lambda)}$ and express I_1 and I_2 in terms of Γ-functions by using k as the variable of integration, (*) follows. To obtain an $f(x)$ giving equality in (*), one secures equality in each application of Hölder's inequality by taking

$$f(x)=y^{\lambda/(p-1)}\{k(yx)\}^{1/(p-1)}x^{-1+\lambda/(p-1)} = y^{-\mu/(q-1)}\{1-k(yx)\}^{1/(q-1)}x^{-1-\mu/(q-1)}.$$

The corresponding series inequality can be proved similarly or derived from (*). Finally, the author remarks that (*) remains true for $p>0$ and $q>0$ if $s+t<1$, but the constant is no longer exact. *R. P. Boas, Jr.* (Providence, R. I.).

Tietze, Heinrich. Der Satz von Rolle als Sonderfall differentialgeometrischer Existenzprobleme. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 77–79 (1947).

This note is concerned with generalizations of the theorem of Rolle. The following statement is an example. Let K be a bounded convex region in the (x, y)-plane, with boundary R. Let $f(x, y)$ be a function defined in K, with the following properties: (i) $f(x, y)$ is continuous in K, (ii) $f(x, y)=0$ on R, (iii) $f(x, y)$ is totally differentiable in $K-R$. Then there exists some point (x_0, y_0) in $K-R$ where $\partial f/\partial x$ and $\partial f/\partial y$ vanish simultaneously. The note contains no proofs.
T. Radó (Columbus, Ohio).

Schoenberg, I. J. On variation-diminishing integral operators of the convolution type. Proc. Nat. Acad. Sci. U. S. A. **34**, 164–169 (1948).

In a previous note [same Proc. **33**, 11–17 (1947); these Rev. **8**, 319] the author introduced the class of Pólya frequency functions $\{\Lambda(x)\}$. The bilateral Laplace transform of $\Lambda(x)$ is the reciprocal of an entire function $\Psi(s)$ of type II in the terminology of Laguerre, Pólya and I. Schur. One of the characteristic properties of $\Lambda(x)$ is that it is totally positive in the sense that $\det\|\Lambda(x_i-t_j)\|\geqq 0$ for any choice of $\{x_i\}$ and $\{t_j\}$, $x_1<\cdots<x_n$, $t_1<\cdots<t_n$. On the other hand, a real matrix $A=\|a_{ij}\|$ is totally positive if all its minors are nonnegative. The author has shown [Math. Z. **32**, 321–328 (1933)] that the corresponding linear transformation $y_i=\sum_{j=1}^n a_{ij}x_j$ $(i=1, \cdots, n)$ is variation-diminishing in the sense that $v(y_i)\leqq v(x_j)$ if $v(x_j)$ and $v(y_i)$ are the number of variations of sign in the sequences $\{x_j\}$ and $\{y_i\}$. In the present note the author studies the transformation

$$g(x)=\int_{-\infty}^{\infty}\Lambda(x-t)f(t)dt,$$

where $\Lambda(x)\in L(-\infty, \infty)$ and $f(t)$ is bounded and measurable. The number of variations of sign of $f(x)$ in $(-\infty, \infty)$ is defined as $v[f]=\sup v[f(x_i)]$ for any choice of n and real numbers x_i, $-\infty<x_1<\cdots<x_n<\infty$. The transformation is said to be variation-diminishing if always $v[g]\leqq v[f]$ and it has this property if and only if either $\Lambda(x)$ or $-\Lambda(x)$ is almost everywhere equal to a Pólya frequency function. A considerable portion of the note is devoted to a discussion of the relations between the investigations of the author and those of T. Motzkin, G. Pólya, and D. V. Widder. The variation theorem has been proved by Pólya (i) in the special case $\Lambda(x)=e^{-x^2}$ [Z. Angew. Math. Mech. **13**, 125–128 (1933)] in connection with the heat equation and (ii) in the general case when $f(x)$ is a polynomial and the convolution operator is replaced by the differential operator $[\Psi(D)]^{-1}$ [J. Reine Angew. Math. **145**, 224–249 (1915)]. In the latter case the inversion formula $f(x)=\Psi(D)g(x)$ holds; the author calls attention to the corresponding inversion formulas for convolution transforms, recently found by Widder [same Proc. **33**, 295–297 (1947); these Rev. **9**, 90].
E. Hille (New Haven, Conn.).

Wright, E. M. Functional inequalities. J. London Math. Soc. **22** (1947), 205–210 (1948).

Let $f(x)$ be real and defined on $0\leqq x\leqq\frac{1}{2}\pi$, and write $a_n=\max|f^{(n)}(x)|$. (1) If $f(x)\in C^*$, if $f(0)=0$, $f(\frac{1}{2}\pi)\leqq 1$ and

$a_n \leqq 1$ for $n \geqq 1$, then $|f(x)| \leqq \sin x$. More general results are as follows. (2) Write $\beta_n = \max\{0, \pm \max_x f^{(n)}(x)\}$, where the upper or lower sign is taken according as $n \equiv 0, 1$ or $n \equiv 2, 3 \pmod 4$. If $f(x) \in C^N$, $N \geqq 2$; $f(0) \leqq 0$, $f(\frac{1}{2}\pi) \leqq 1$; $(-1)^{\frac{1}{2}(n-1)} f^{(n)}(0) \leqq 1$ for odd n and $(-1)^{\frac{1}{2}n} f^{(n)}(\frac{1}{2}\pi) \leqq 1$ for even n, $1 \leqq n \leqq N$; and $\beta_{N-1}^2 + \beta_N^2 \leqq 1$, then $f(x) \leqq \sin x$. (3) If $f(x) \in C^\infty$; $f(0) \leqq 0$; $(-1)^{\frac{1}{2}(n-1)} f^{(n)}(0) \leqq 1$ for odd n; for some $\delta > 0$ there is a function $\lambda(x)$ such that, for $\frac{1}{2}\pi - \delta < x < \frac{1}{2}\pi$, $0 < \lambda(x) < 1$, $f(x) \leqq 1$, $(-1)^{\frac{1}{2}n} f^{(n)}(x) \leqq \{\frac{1}{2}\pi\lambda(x)/x\}^n$ for even n; and $\liminf n^{-1} \log a_n \leqq 0$, then $f(x) \leqq \sin x$.

R. P. Boas, Jr. (Providence, R. I.).

Obreškov, N. On certain inequalities involving differences of a function of a real variable. Doklady Akad. Nauk SSSR (N.S.) **59**, 1399–1401 (1948). (Russian)

Let $\varphi(x)$ and $\psi(x)$ be real functions, continuous for $x > a$, and let $\varphi(y_\lambda) y_\lambda^{-m} \to B$, $\psi(y_\lambda) y_\lambda^{-m} \to C$ as $y_\lambda \to \infty$, for some sequence $\{y_\lambda\}$. For an integer $n > m$ let, for $x > a$ and some $h > 0$, $\Delta_h^n \psi(x) \leqq \Delta_h^n \varphi(x)$. Let $\omega(f; x_0, \cdots, x_m)$ denote the divided difference of $f(x)$ based on the points x_i. If all $x_i > a$, then

$$(-1)^{n-m}[\omega(\psi; x_0, \cdots, x_m) - C] \leqq (-1)^{n-m}[\omega(\varphi; x_0, \cdots, x_m) - B].$$

This extends earlier results of the author dealing with derivatives and ordinary differences. [Cf. C. R. Acad. Sci. Paris **224**, 880–882 (1947); these Rev. **8**, 448, 709.]

R. P. Boas, Jr. (Providence, R. I.).

Giorgi, Giovanni. Formole per la derivazione a indice generalizzato. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 693–701 (1942).

A critical review of the theory of differentiation of arbitrary order. *I. Opatowski* (Ann Arbor, Mich.).

Mandelbrojt, S. Analytic continuation and infinitely differentiable functions. Bull. Amer. Math. Soc. **54**, 239–248 (1948).

This expository article gives a clear account of the meaning of the author's "fundamental inequality" [Ann. Sci. École Norm. Sup. (3) **43**, 351–378 (1946); these Rev. **9**, 229] and of its applications. Proofs are only sketched. The applications include many theorems on the singularities and on the analytic continuation of functions defined by Dirichlet series and generalisations of the author's earlier results on quasi-analytic functions [Trans. Amer. Math. Soc. **55**, 96–131 (1944); these Rev. **5**, 176]. An example of these applications is the following theorem. Let $N(x)$ be the number of λ_n less than x ($\lambda_0 > 0$; $\lambda_{n+1} - \lambda_n > h > 0$), $\bar{D} = \limsup_{\lambda \to \infty} \lambda^{-1} \int_0^\lambda \{N(x)/x\} dx$. If $\sum d_n e^{-\lambda_n s}$ represents an integral function $F(s)$ of order ρ which is not a Dirichlet polynomial, then each strip wider than $2\pi \max(\bar{D}, (2\rho)^{-1})$ contains a line $t = t_0$ such that $F(s)$ takes every value with at most one exception infinitely often in $|t - t_0| < \epsilon$ for every $\epsilon > 0$. It is pointed out that the "fundamental inequality" is near to being a best possible result. *W. H. J. Fuchs.*

Brunk, Hugh. Conditions de convergence pour des séries asymptotiques. C. R. Acad. Sci. Paris **226**, 460–462 (1948).

Following Mandelbrojt we say that $\sum d_k e^{-\lambda_k s}$ represents $F(s)$ for a given n with logarithmic precision $p(x)$ in the domain Δ of the $(\sigma + it)$-plane, if

$$\inf_{m \geqq n} \left(\sup_{\sigma \geqq x, s \in \Delta} |F(s) - \sum_{k=1}^{m} d_k e^{-\lambda_k s}| \right) \leqq e^{-p(x)}.$$

In this paper $\{\lambda_n\}$ is an increasing sequence of positive numbers satisfying (i) $\lambda_{n+1} - \lambda_n > c > 0$; (ii) if $N(x) = \sum_{\lambda < x} 1$, $\bar{D}(x) = \sup_{x \geqq v} v^{-1} \int_0^v N(v) dv$, then $\bar{D} = \lim_{x \to \infty} \bar{D}(x) < \infty$. Theorem I. Hypothesis 1: $F(s)$ is regular in the domain $\Delta: \sigma > b$, $|t| < \pi g(\sigma)$, where $g(\sigma)$ is of bounded variation for $\sigma > b$ and $\lim_{\sigma \to \infty} g(\sigma) > \bar{D}$. Hypothesis 2: for an infinity of n the series $\sum d_k e^{-\lambda_k s}$ represents $F(s)$ with logarithmic precision $p_n(x)$ and

$$(*) \qquad \int^\infty p_n(\sigma) \exp\left\{ -\tfrac{1}{2} \int^\sigma (g(u) - \bar{D}(p_n(u)))^{-1} du \right\} d\sigma = \infty.$$

Conclusion: $\sum d_k e^{-\lambda_k s}$ converges to $F(s)$ or its analytic continuation for $\sigma > \sigma_0$. The convergence of $\phi(s) = \sum d_k e^{-\lambda_k s}$ is a consequence of Mandelbrojt's "fundamental inequality" [cf. the preceding review]. The author deduces $\phi(s) = F(s)$ by establishing a homogeneous linear differential equation for the difference of the two functions and applying a method of Ritt [Trans. Amer. Math. Soc. **18**, 21–26, 27–49 (1917)]. The following application of theorem I is given. Suppose $A_n(\sigma) \downarrow 0$, $A_{n+1}(\sigma) = O(A_n(\sigma))$ as $\sigma \to \infty$. Put $A(\sigma) = \inf_n A_n(\sigma)$. Theorem II. If hypothesis 1 of theorem I is satisfied, if

$$|F(s) - \sum_{k=1}^{n} d_k e^{-\lambda_k s}| < A_n(\sigma), \qquad \sigma > \sigma_0,\ n \geqq 1,$$

and if (*) holds with $p_n(\sigma) = -\log A(\sigma)$, then the conclusion of theorem I holds. *W. H. J. Fuchs* (Ithaca, N. Y.).

Fedorov, V. S. On a property of curvilinear integrals. Doklady Akad. Nauk SSSR (N.S.) **58**, 755–756 (1947). (Russian)

Let $P(x, y)$ and $Q(x, y)$ be continuous in a finite simply connected domain D; let J be a Jordan region of area a and rectifiable boundary L in D; let $\{J_n\}$ be a sequence of Jordan regions with rectifiable boundaries c_n, each including the origin, of area a_n, and with diameter approaching 0 as $n \to \infty$; and let $c_n(x, y)$ be the image of c_n under parallel displacement with the origin mapped on (x, y). The author announces a result approximating the value of $I(L)/a$, where

$$I(L) = \oint_L P dx + Q dy,$$

in terms of the sequence of values $I[c_n(x_n, y_n)]/a_n$.

The above result affords a ready tool for establishing certain theorems which improve familiar results, in that hypotheses in the large are replaced by apparently weaker point conditions. Thus in the theory of analytic functions of a complex variable we have the following familiar improvement of the theorem of Morera. If $f(z)$ is single-valued and continuous in the interior of D, and there exists a sequence $\{c_n\}$ such that for each point ζ in D and for each sequence $\zeta_n \to \zeta$ such that ζ is contained in $c_n(\zeta_n)$ we have

$$\lim_{n \to \infty} a_n^{-1} \oint_{c_n(\zeta_n)} f(z) dz = 0,$$

then $f(z)$ is analytic in D. *E. F. Beckenbach.*

Theory of Sets, Theory of Functions of Real Variables

Johnston, L. S. Denumerability of the rational number system. Amer. Math. Monthly **55**, 65–70 (1948).

In this paper the author exhibits a counting system by means of which one may find the ordinal of a given rational

number and the rational number corresponding to a given ordinal. The method depends upon certain properties of an array with elements $[m, n]$, where

$$[m, n]=(mx+ny)/(my+nx).$$

From the properties of this array the author develops an algorithm, suggestive of the Euclidean algorithm on highest common factor, for obtaining the ordinal $Q(m, n)$ of a proper fraction $n/(m+n)$ in its lowest terms, and for computing m, n when $Q(m, n)$ is given. By means of a second array, whose elements are counted in the Cantor order, he then obtains the ordinal number, in terms of p and Q, for $p+n/(m+n)$, $(p=0, 1, 2, \cdots)$, the set of all positive rationals, including zero. The ordering is easily extended to include negative rationals. Numerical examples illustrate the methods. *W. H. Gage* (Vancouver, B. C.).

Bagemihl, Frederick. Some theorems on powers of cardinal numbers. Ann. of Math. (2) **49**, 341–346 (1948).

Several theorems are established which show that, given a cardinal $\mathfrak{c}$, certain kinds of transfinite cardinals cannot be represented as powers of $\mathfrak{c}$ and certain kinds can possibly be only small powers of $\mathfrak{c}$. For instance, it is shown that if $\mathfrak{b}$ is a cardinal and S an increasing sequence such that to every term there corresponds a greater term which is a power of $\mathfrak{b}$ then the limit $\mathfrak{a}$ of S is not a power of any $\mathfrak{c}\leqq\mathfrak{b}$. For example, if $\mathfrak{b}=2$ and S is $\{1, 2, 2^2, \cdots, 2^n, \cdots\}$ we have $\mathfrak{a}=\aleph_0$ which is not a power of 2. *J. Todd* (London).

Lyapunov, A. A. A new definition of certain classes of sets. Doklady Akad. Nauk SSSR (N.S.) **59**, 847–848 (1948). (Russian)

The author presents a class of operations on families of sets, which is not described in detail. He announces that analytic sets in the line are characterized as the family of sets obtained from intervals by all possible operations of this class. *E. Hewitt* (Seattle, Wash.).

Ščegol'kov, E. A. On the uniformization of certain B-sets. Doklady Akad. Nauk SSSR (N.S.) **59**, 1065–1068 (1948). (Russian)

The following theorem is proved. Let A be any Borel set in the plane such that $A\cap E[(x, y); x=x_0]$ is an F_σ for all x_0. Then there exists a subset C of A such that C is a Borel set, $C\cap E[(x, y); x=x_0]$ contains at most one point, and C and A have identical projections on the X-axis. *E. Hewitt* (Seattle, Wash.).

Barbalat, B. Sur les fonctions monotones. Bull. Math. Soc. Roumaine Sci. **47**, 62–65 (1946).

The function $f(x)$ is defined and bounded on $[a, b]$; $M[a, x]$, $m[a, x]$ are the supremum and infimum, respectively, of $f(x)$ on $[a, x]$; $V[a, x]$ is the total variation, $\omega[a, x]$ the oscillation of $f(x)$ on $[a, x]$. If $f(x)$ is monotone, then $V[a, x]=|f(x)-f(a)|=\omega[a, x]$, $\omega[a, b]=\omega[a, x]+\omega[x, b]$. It is shown that if these conditions are satisfied then $f(x)$ is monotone. *R. L. Jeffery* (Kingston, Ont.).

Marczewski, E., et Nosarzewska, M. Sur la convergence uniforme et la mesurabilité relative. Colloquium Math. **1**, 15–18 (1947).

La simple observation suivante: "La limite d'une suite $f_1(x), f_2(x), \cdots$ de fonctions monotones dans l'intervalle $a\leqq x\leqq b$ est continue, si et seulement si la suite converge uniformément et, en chaque point de l'intervalle $a\leqq x\leqq b$, l'oscillation de $f_n(x)$ tend vers zéro, lorsque n tend vers $+\infty$" entraîne immédiatement une condition nécessaire et suffisante pour la continuité de la distributrice, ou distribuante [Kac et Steinhaus, Studia Math. **7**, 1–15 (1938), p. 4], d'une fonction $f(x)$ définie pour $x\geqq 0$ et relativement mesurable. *G. Scorza Dragoni* (Padova).

Jessen, Børge. Abstract theory of measure and integration. VIII. Mat. Tidsskr. B. **1947**, 1–20 (1947). (Danish)

Jessen, Børge. Abstract theory of measure and integration. IX. Mat. Tidsskr. B. **1947**, 21–26 (1947). (Danish)

Jessen, Børge. Abstract theory of measure and integration. X. Mat. Tidsskr. B. **1947**, 27–31 (1947). (Danish)

Continuation and conclusion of a series of expository articles. The subtitles to the three parts are, respectively: Orthonormal systems; The ergodic theorem; The calculus of probability. [The previous instalments appeared in the same Tidsskr. B in 1934, 1935, 1938, 1939, 1942, 1944; cf. these Rev. **1**, 110; **7**, 197. Cf. also the following review.] *W. Feller* (Ithaca, N. Y.).

* **Jessen, Børge. Abstrakt Maal- og Integralteori. [Abstract Theory of Measure and Integration.]** Matematisk Forening i København, København, 1947. iii+108 pp. (Danish)

This book collects the series of articles of which the last three are reviewed above.

Mambriani, Antonio. Sul problema di Geöcze. Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 1–17 (1948).

Let $L(S)$ be the Lebesgue area of a continuous surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $0\leqq u\leqq 1$, $0\leqq v\leqq 1$. Let $L^*(S)$ be the quantity obtained by restricting the polyhedra occurring in the definition of $L(S)$ to polyhedra inscribed in S. Clearly $L^*(S)\geqq L(S)$. The problem of Geöcze is the problem to decide whether the sign of equality holds. The author proves that a sufficient condition is the existence of a representation with the following properties. (i) The coordinate functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are absolutely continuous in the Tonelli sense. (ii) The first partial derivatives $x_u, \cdots, z_v$ are summable L^2. Let now S be a surface given in the nonparametric form $z=f(x, y)$, $0\leqq x\leqq 1$, $0\leqq y\leqq 1$. If $L(S)<+\infty$, then by a result of Morrey the surface S possesses a representation with the properties (i), (ii). Thus the problem of Geöcze is solved for the nonparametric case. In previous work on the Geöcze problem in the nonparametric case, the quantity $L^*(S)$ was interpreted in terms of inscribed polyhedra which also admit of nonparametric representations $z=f(x, y)$, with $f(x, y)$ continuous and single-valued. It may be of interest to determine if the results of the present paper account for the problem of Geöcze in this more exacting form. *T. Radó* (Columbus, Ohio).

Helsel, R. G., and Mickle, E. J. The Kolmogoroff principle for the Lebesgue area. Bull. Amer. Math. Soc. **54**, 235–238 (1948).

Kolmogoroff [Math. Ann. **107**, 351–366 (1933)] used as an axiom in measure theory the principle that if a transformation does not increase any of the distances between the points of a set then it must not increase the measure of the set. The authors show that the Lebesgue area of a Fréchet surface satisfies a version of this principle, as

follows. Let T and T^* be two continuous transformations from the unit disc $D: u^2+v^2 \leqq 1$ into Euclidean 3-space, and let $A(T)$ and $A(T^*)$ be the Lebesgue areas of the Fréchet surfaces determined by T and T^*, respectively. Then, if the distance inequality $|T^*(p_1)-T^*(p_2)| \leqq |T(p_1)-T(p_2)|$ holds for every pair of points $p_1, p_2 \in D$, we have $A(T^*) \leqq A(T)$.

H. P. Mulholland (Beirut).

Mickle, Earl J. Metric foundations of continuous transformations. Trans. Amer. Math. Soc. 63, 368–391 (1948).

Let $T: p^* = t(p)$, $p \in A \subset M$, $p^* \in A^* \subset M^*$, be a single-valued, continuous transformation from an analytic set A in the complete and separable metric space M onto a set A^* in the metric space M^*. For each point p^* and a given set $E \subset M$, the counting function $N(p^*, T, E)$ is defined to be the number of distinct points in the set $E \cdot T^{-1}(p^*)$. Let A_∞^* be the set of those points p^* for which $N(p^*, T, A) = +\infty$ and let $A_\infty = T^{-1}(A_\infty^*)$. The author first shows that, if $\nu^*(E^*)$ is a finite Carathéodory outer measure defined on M^* which is regular with respect to open sets, then there exists a completely additive class of sets in M which contains all analytic sets and for which the set function $\mu(E, T) = \int_{M^*} N(p^*, T, E - A_\infty) d\nu^*$ is a measure.

If $f(p)$ is any finite-valued function on A and $p^* \overline{\in} A_\infty^*$, then, for a given set $E \subset M$ such that $E \cdot T^{-1}(p^*) \neq 0$, the function $\sigma(p^*, T, E, f)$ is defined to be the sum of the values of the function $f(p)$ taken at the points of the finite set $E \cdot T^{-1}(p^*)$. If $E \cdot T^{-1}(p^*) = 0$ or $p^* \in A_\infty^*$, then $\sigma(p^*, T, E, f)$ is defined to be zero. The author shows that if $H(p^*)$ is a finite-valued ν^*-measurable function on M^* and $f(p)$ is any finite-valued μ-measurable function on M, then, for any μ-measurable set $E \subset A$, the fundamental transformation formula $\int_E H[t(p)] f(p) d\mu = \int_{M^*} H(p^*) \sigma(p^*, T, E, f) d\nu^*$ holds if the integral on the left exists. If $f(p) \geqq 0$, the formula holds whenever the integral on the right exists. In case the counting function $N(p^*, T, A)$ is ν^*-summable on M^*, the transformation T is said to be of bounded variation on A and the preceding formula implies that $\int_E H[t(p)] d\mu = \int_{M^*} H(p^*) N(p^*, T, E) d\nu^*$, provided one of the integrals exists.

Let $\nu(E)$ be a finite Carathéodory outer measure defined on M which is regular with respect to open sets. If the transformation T is of bounded variation on A and if $\nu^*[T(E)] = 0$ for every set $E \subset A$ of ν-measure zero, then T is said to be absolutely continuous on A. In case T is absolutely continuous on A, $\mu(E, T)$ is an absolutely continuous completely additive set function on the class of ν-measurable sets and it follows from the Radon-Nikodym theorem that $\mu(E, T)$ can be expressed as the ν-integral of a nonnegative finite-valued ν-measurable function $D(p, T)$. Assuming that T is absolutely continuous on A, it is shown that, if $f(p)$ is a finite-valued, ν-measurable function on M and $H(p^*)$ is a finite-valued ν^*-measurable function on M, then, for any ν-measurable set $E \subset A$, $\int_E H[t(p)] f(p) D(p, T) d\nu = \int_{M^*} H(p^*) \sigma(p^*, T, E, f) d\nu^*$ whenever the left integral exists and, if $f(p) \geqq 0$, whenever the right integral exists. In addition, the formula $\int_E H[t(p)] D(p, T) d\nu = \int_{M^*} H(p^*) N(p^*, T, E) d\nu^*$ holds if either integral exists.

To test the applicability of his results, the author shows that certain useful formulas derived by Radó and Reichelderfer [same Trans. 49, 258–307 (1941); these Rev. 2, 257] for plane transformations are special cases of the general formulas described above.

R. G. Helsel.

Trjitzinsky, W. J. Theory of functions of intervals and applications to functions of a complex variable. J. Math. Pures Appl. (9) 25 (1946), 347–395 (1947).

The author generalizes theorems of Ridder and Besicovitch concerning functions of intervals in Euclidean n-space R_n and applies them to problems of analyticity and to problems of representations of functions of a complex variable. Ridder's theorem, in which the following set D_0 was assumed to be a denumerable set of points, is generalized thus: if N is a normal sequence of nets, $g(x)$ is summable in R_n, F is a continuous additive interval function such that the lower derivate of F relative to N satisfies $(N)\underline{F}(x) > -\infty$ at all points x except those of a set D_0 of points of a denumerable infinitude of hyperplanes parallel to one of the axes, and we have $F'(x) \geqq g(x)$ almost everywhere, then for intervals I we have $F(I) \geqq \int_I g(x) dx$. The extension of the theorem of Besicovitch is based on a new definition of the μ-length of a set, relative to any function $\mu(u)$ which satisfies $\mu(u) > 0$ for $u > 0$ and $\mu(u) \to 0$ monotonically as $u \to 0$; the definition reduces to that of Besicovitch for $\mu(u) = u^\alpha$, $\alpha > 0$.

Applications to complex variable theory include extensions of results of Wolff and Besicovitch concerning conditions of analyticity. Thus it is shown that $f(z)$ is analytic in a domain D provided we have in D

$$\liminf |I|^{-1} \left| \int_I f(z) dz \right| = 0, \qquad I \supset z_0,$$

for almost all z_0, and

$$\limsup |I|^{-1} \left| \int_I f(z) dz \right| < +\infty, \qquad I \supset z_0,$$

except at most at points z_0 of a set E consisting of a denumerable infinitude of rectilinear segments parallel to the axes.

E. F. Beckenbach (Los Angeles, Calif.).

Rechard, O. W., and Reichelderfer, P. V. A new criterion for the extension of rectangle functions. Duke Math. J. 15, 151–158 (1948).

Let $\mathbf{R}$ be a fixed oriented closed rectangle in the (x, y)-plane, and let $\mathfrak{R}$ be the class of all oriented closed non-degenerate rectangles contained in $\mathbf{R}$. Let ϕ be a function with domain $\mathfrak{R}$ and range in the set of nonnegative, finite real numbers. Theorem: ϕ admits a completely additive extension to an additive class of sets including all Borel sets if and only if it satisfies the following three conditions: (i) if R_1 and R_2 are two mutually exclusive rectangles in $\mathfrak{R}$ and R is any rectangle in $\mathfrak{R}$ such that $R_1 + R_2 \subset R$, then $\phi(R_1) + \phi(R_2) \leqq \phi(R)$; (ii) if R is any rectangle in $\mathfrak{R}$ and R_1 and R_2 are any two rectangles in $\mathfrak{R}$ such that $R \subset R_1 + R_2$, then $\phi(R) \leqq \phi(R_1) + \phi(R_2)$; (iii) if R is any rectangle in $\mathfrak{R}$ and ϵ is a positive number, let R_ϵ denote the common part of $\mathbf{R}$ and the rectangle containing R with sides parallel to the corresponding sides of R at distance ϵ. Then $\lim \phi(R_\epsilon) = \phi(R)$ as ϵ tends to zero through positive values. The conditions are clearly necessary, and sufficiency is established by showing that (i), (ii) and (iii) imply (C) [stated below] which is known [cf. same J. 8, 231–242 (1942); these Rev. 3, 74] to imply the conclusion. (C) If $R_1, R_2, \cdots$ is a finite or denumerable sequence of mutually exclusive rectangles in $\mathfrak{R}$, and if $R_1^*, R_2^*, \cdots$ is any finite or denumerable sequence of rectangles in R such that $\sum R_n \subset \sum R_n^*$, then $\sum \phi(R_n) \leqq \sum \phi(R_n^*)$. Similar necessary and sufficient conditions for extensions are established for open rectangles and for nonoriented rectangles.

J. F. Randolph (Rochester, N. Y.).

Maximoff, Isaiah. On continuous transformation of some functions into an ordinary derivative. Ann. Scuola Norm. Super. Pisa (2) 12 (1943), 147–160 (1947).

The author solves the following problem proposed to him by N. Lusin. Let $f(x)$ be a finite function of class 1 having the Darboux property in $[0, 1]$ (that is, attaining every value between any two values of $f(x)$). Find a continuous and essentially increasing function $x=\varphi(t)$ with $\varphi(0)=0$, $\varphi(1)=1$, such that $f[\varphi(t)]$ is the derivative of a continuous function in $0\leqq t\leqq 1$. The affirmative solution given by the author consists in a (rather involved) construction of a function $x=\varphi(t)$ such that $f[\varphi(t)]$ is approximately (ρ_n, Δ_n) continuous in $0\leqq t\leqq 1$. This notion was previously defined by the author [Bull. Sci. Math. (2) 64, 116–121 (1940); these Rev. 4, 5]. Then from the main result of the same previous paper it follows that $f[\varphi(t)]$ is the derivative of a continuous function in $0\leqq t\leqq 1$. *A. Rosenthal.*

Choquet, Gustave. Ensembles singuliers et structure des ensembles mesurables pour les mesures de Hausdorff. Bull. Soc. Math. France 74, 1–14 (1946).

Let C be the linear measure of Carathéodory in the plane. If $2^{\aleph_0}=\aleph_1$, then there exists a set S with the following properties: (1) $C(S)=\infty$, (2) every subset of S is C-measurable, (3) if $T\subset S$, then either $C(T)=0$ or $C(T)=\infty$, (4) every analytic subset of S is countable. *H. Federer.*

Choquet, Gustave. Application des propriétés descriptives de la fonction contingent à la théorie des fonctions de variable réelle et à la géométrie différentielle des variétés cartésiennes. J. Math. Pures Appl. (9) 26 (1947), 115–226 (1948).

A mapping f of a Hausdorff space E in a Hausdorff space E' is called pointwise discontinuous on a set $P\subset E$ if the contraction of f to P is continuous at at least one point. The mapping f is of the first Borel class if $f^{-1}(G)$ is an F_σ for every open set $G\subset E'$. Functions of higher class are defined from these and the continuous functions by the usual limit processes. If E is locally compact and E' is separable and metric, then every mapping of the first class is pointwise discontinuous on every closed subset of E. The converse is in general not true. Let f be a mapping of E on the space formed by the closed subsets of E'. Then f is upper semicontinuous if for every point $p\varepsilon E$ and every neighborhood $V(f(p))$ a neighborhood $U(p)$ exists such that $p_1\varepsilon U(p)$ implies $f(p_1)\subset V(f(p))$. If for every point $p'\varepsilon f(p)$ and every $V(p')$ a $U(p)$ exists such that $p_1\varepsilon U(p)$ implies $f(p_1)\cap V(p')\neq 0$, then f is lower semicontinuous. If E is compact and E' is metric then any upper or lower semicontinuous f is pointwise discontinuous on every perfect set in E.

Let the set Θ of the unit vectors in the n-dimensional Euclidean space R_n be metrized by the angle, and let 2^Θ be the set of all closed subsets of Θ metrized by the Hausdorff distance. For a closed set E in R_n the set of directions $C(p)$ of the contingent of E at a point p lies in 2^Θ, and $C(p)$ is at most of the second class. Actually $C(p)$ is the limit of a decreasing sequence of lower semicontinuous functions. If T is a continuous mapping of a Hausdorff space E in R_n, then the contingent of T at a point $p\varepsilon E$ is the accumulation set of all rays $T(p)T(p')$, if p' approaches p without passing points with $T(p)=T(p')$. There are various theorems on the case where $T(p)$ is higher dimensional. If E is compact and the contingent of T is empty or carried by a straight line of direction $g(p)$ then the set F of points in E at which $g(p)$ is determined is closed and $g(p)$ is pointwise discontinuous on F. If the contingent of T at p is a whole line then T is said to have a tangent (vraie tangente) at p. If the contingent is a ray, T has a cusp at p (tangente de rebroussement; an endpoint of a differentiable arc is a cusp). Let T be a continuous mapping of a compact connected space E in the plane, which has everywhere a tangent or a cusp of direction $g(p)$. If T has at most one cusp then $g(p)$ traverses straight lines of all directions. If T has two cusps a, b, then a point $p\varepsilon E$ exists at which $g(p)$ is parallel to the line through $T(c)T(b)$ (mean value theorem).

Let $f(p)$ and $\alpha(p)$ be continuous real-valued functions on a Hausdorff space E. Then $f(p)$ has at p a derivative $g(p)$ with respect to α ($+\infty$ and $-\infty$ admitted but considered as the same number) if $[f(p')-f(p)]/[\alpha(p')-\alpha(p)]\to g(p)$ when p' approaches p so that numerator and denominator do not both vanish. We associate with $f(p)$ and $\alpha(p)$ the point $x=\alpha(p)$, $y=f(p)$ of the (x, y)-plane. Then $g(p)$ exists if the contingent of the thus defined mapping T of E on the plane at p is not empty and is carried by a line. If E is compact and locally connected then any two continuous functions on E which have at every point p the same derivative $g(p)$ with respect to $\alpha(p)$ differ only by a constant. The primitive function can be obtained from $g(p)$ (and $\alpha(p)$) by a process similar to Denjoy's totalization.

Denote by I the interval $0<t<1$. The continuous mapping T of I in R_n is regular at t_0 if a straight line L_0 through $T(t_0)$ exists such that to every open interval $v\supset t_0$ an open interval $V\supset T(t_0)$ on L_0 can be found for which the Fréchet distance ϵ of V and $T(v)$ satisfies the relation $\lim \epsilon/\delta=0$, where δ is the diameter of V. Two mappings T, T' of I are equivalent if a topological mapping $p'(p)$ of I on itself exists with $T(p)=T'(p'(p))$. The mapping T is (positively) differentiable if the vector derivative, that is, $d\overrightarrow{OT(t)}/dt$ exists (and is different from 0). One of the principal results of the paper is the following answer to a problem of Fréchet. (*) The mapping $T(t)$ is equivalent to an everywhere positively differentiable mapping if and only if T is everywhere regular.

A point p of a simple arc γ of R_n is regular if the identical mapping T of γ on itself is regular at p in the above sense. In answer to a question of Pauc it is shown that simple arcs in R_n exist which have everywhere a tangent, but cannot be parameterized in such a way that they are positively differentiable at every regular point.

The mapping $T(t)$ is called partially rectifiable if the following holds for every perfect subset P of I. There is an open interval D containing points of P and (1) T has finite linear measure on $D\cap P$; (2) if δ_i is the diameter of the image $T(\sigma_i)$ of an interval σ_i contiguous to $D\cap P$ then $\sum\delta_i$ is finite. A necessary and sufficient condition for $T(t)$ to be equivalent to a differentiable mapping is that T is partially rectifiable.

The concepts of regularity, differentiability and positive differentiability are extended from curves, that is, mappings of the interval I, to surfaces, that is, mappings of domains E of R_p in R_n. It is shown that the main result (*) still holds.

Let $g(x)$ be a function of the first class on the closed segment $(0, 1)$. The graph of $g(x)$ is connected if and only if $g(x)$ has the property of Darboux. Various properties of semicontinuous functions are obtained which are equivalent to Darboux's property. Another main result is the following. If $g(x)$ is lower semicontinuous in $(0, 1)$, has the Darboux property and is bounded, then it is topologically equivalent to a derivative, that is, a topological mapping $x(y)$ of $(0, 1)$

on itself exists such that $f(y)=g(x(y))$ is a derivative. Finally, theorems on the structure of sets where a derivative has a fixed value are proved, for instance: a set in (0, 1) is the set of points where a derivative is $+\infty$ if and only if it is a G_δ of measure 0. *H. Busemann.*

Theory of Functions of Complex Variables

Ghika, Al. Sur certains espaces fonctionnels de fonctions analytiques. Bull. Math. Soc. Roumaine Sci. **47**, 10–28 (1946).

Let L be a rectifiable curve bounding a simply connected domain D. Let $C_p(D)$ be the class of functions $f(z)$ regular in D, having boundary values whose pth powers are integrable on L, and representable by the Cauchy integral of these boundary values in D. Let $H_p(D)$ be the class of functions $f(z)$ regular in D and of bounded pth means on a sequence of polygons approximating L arbitrarily closely in D. The author shows that $H_p(D)\subset C_p(D)$ if $p\geqq 1$ and that $H_p(D)\equiv C_p(D)$ for all $p\geqq 1$ if and only if $\log|\phi'(z)|$ is expressible as the Poisson integral of its boundary values on $|z|=1$, where $\phi(z)$ maps $|z|<1$ conformally on D. *W. K. Hayman* (Exeter).

Wolf, František. Extension of analytic functions. Duke Math. J. **14**, 877–887 (1947).

The following result has been established by T. Carleman. If $f_1(z)$ and $f_2(z)$ are analytic in the upper half and the lower half, respectively, of the unit circle Q, and (1) $\lim_{y\to+0}[f_1(x+iy)-f_2(x-iy)]=0$ uniformly for $|x|<1$, then f_1 and f_2 are parts of the same analytic function f in Q. The author investigates the extent to which the conclusion of this result remains valid when the hypothesis of uniformity is dropped. It is easy to show that the conclusion still holds when boundedness is substituted for uniformity. If the hypothesis of uniformity is dropped altogether, then f_1 and f_2 are parts of the same function f which is analytic on an everywhere dense set of points of $|x|<1$, $y=0$. For example, the function $1/z^2$ satisfies the present hypothesis. But (1) and the uniformity of $f_1(x+iy)-f_2(x-iy)=o(y^{-1})$ as $y\to+0$ in Q imply the analyticity of f throughout Q. [We note that the last formula on p. 882 is necessarily correct only if the isolated point in question does not belong to the exceptional set T, and that in theorem E we must require that T is null; thus the functions $f_1(z)=f_2(z)=1/z$ satisfy the hypotheses of theorem E as stated without satisfying the conclusion of that theorem.] *E. F. Beckenbach* (Los Angeles, Calif.).

Hornich, Hans. Der Schlichtheitsradius bei ganzen Funktionen. Akad. Wiss. Wien, S.-B. IIa. **154**, 59–65 (1945).

Let $f(z)$ be a nonlinear entire function. For each point a for which $f'(a)\neq 0$, $f(z)$ is schlicht for $|z-a|<\rho(a)$, where $\rho(a)=\inf_{z_1,z_2}[\max|z_1-a|,\ |z_2-a|]$ taken for point pairs z_1, z_2, $z_1\neq z_2$, for which $f(z_1)=f(z_2)$. For each a with $\rho(a)>0$ either there is a point $\bar z$, $|\bar z-a|=\rho(a)$, for which $f'(\bar z)=0$ and $f''(\bar z)\neq 0$, or there exists a pair of points $\bar z_1\neq\bar z_2$ with $f(\bar z_1)=f(\bar z_2)$ and $|\bar z_1-a|=|\bar z_2-a|=\rho(a)$. In the latter case $f'(\bar z_1)\neq 0$, $f'(\bar z_2)\neq 0$, and

$$\Im\log f'(\bar z_1)/f'(\bar z_2)=\Im\log(\bar z_2-a)/(\bar z_1-a)+\pi.$$

If, for all sufficiently large values of z, $\rho(z)\geqq c|z|^{-k}$, $c>0$, $k\geqq 0$, then $f(z)$ must have the form

$$f(z)=\int^z e^{P(z)}(z-\alpha_1)\cdots(z-\alpha_n)dz,$$

where $P(z)$ is a polynomial of degree not exceeding $2+2k$. If, for all z of a neighborhood U of a, $\rho(z)\geqq\rho(a)>0$, then $f(z)$ is a periodic function. If $f(z)=R(z)e^{P(z)}$, $P(z)=a_0+a_1z+\cdots+a_nz^n$ ($a_n\neq 0$, $n>0$), $R(z)$ a polynomial, then for each $\epsilon>0$,

$$n-\epsilon<n|a_nz^{n-1}|\rho(z)<2n+\epsilon$$

for all sufficiently large z. Conversely, if $f(z)$ has but a finite number of zeros $\beta_1,\cdots,\beta_k$ and if, for each $\epsilon>0$ and sufficiently large z, $a-\epsilon<|z|^{n-1}\rho(z)<2a+\epsilon$ ($a>0$, n a positive integer), then $f(z)$ must have the form $f(z)=(z-\beta_1)\cdots(z-\beta_k)e^{P(z)}$, where $P(z)$ is a polynomial of degree n. *M. S. Robertson* (New Brunswick, N. J.).

Mandelbrojt, Szolem, et Ulrich, Floyd. Sur les domaines de comportement uniforme d'une fonction analytique. C. R. Acad. Sci. Paris **226**, 152–153 (1948).

"Comportement uniforme" of a family of functions $f(z)$ in $|z|<1$ means that $|f(z_1)|<A(\theta)|f(z_2)|$ or $|\log|f(z_1)||<A(\theta)|\log|f(z_2)||$ for any $|z_1|<\theta$, $|z_2|<\theta$, $0<\theta<1$. It is equivalent to normality of the family [see S. Mandelbrojt, J. Math. Pures Appl. (9) **8**, 173–195 (1929)]. It is shown that if in addition $|f'(0)|>k|f(0)\log f(0)|$, $\log|f(0)|>k$, $f'(z)\neq 0$, then $f'(z)$ also has "comportement uniforme." Applications to "flat regions" of analytic functions. *A. J. Macintyre* (Aberdeen).

Hayman, W. K. Remarks on Ahlfors' distortion theorem. Quart. J. Math., Oxford Ser. **19**, 33–53 (1948).

Soit Ω un ensemble ouvert du plan $s=u+iv$ coupé par toute ligne $u=\text{constante}=\sigma$ suivant un segment au plus, θ_σ, de longueur $\theta(\sigma)$. On représente conformément Ω sur la bande $0<\eta<a$, du plan $\zeta=\xi+i\eta$. On pose

$$I=\int_{\sigma_1}^{\sigma_2}[\theta(\sigma)]^{-1}d\sigma,\quad \xi_1(\sigma)=\inf_{s\to\sigma}\xi(s),\quad \xi_2(\sigma)=\sup_{s\to\sigma}\xi(s).$$

L'auteur démontre l'inégalité $\xi_1(\sigma_2)-\xi_2(\sigma_1)\geqq a\xi_0(I)$ ($\sigma_2>\sigma_1$) où $\xi_0(k)$ est la différence des abscisses des images des points B et D lorsque'on représente le rectangle $ABCD$ de côtés $AB=CD=k$, $AD=BC=1$ sur la bande $0<\eta<1$ de manière que le point A corresponde à $\xi=-\infty$ et C à $\xi=+\infty$. Cette inégalité est la meilleure possible, comme le montre l'exemple du rectangle, et complète l'inégalité d'Ahlfors [Acta Soc. Sci. Fennicae. Nova Ser. A. 1, no. 9 (1930)]. L'auteur étudie de la même façon les représentations conformes de Ω sur les domaines contenus dans $|z|<1$, puis les propriétés des fonctions introduites, et leurs conséquences. *J. Lelong Ferrand* (Caen).

Lohwater, A. J., and Seidel, W. An example in conformal mapping. Duke Math. J. **15**, 137–143 (1948).

An example is given of a Jordan curve C and a set of points E on C with the following property. If the interior of C is mapped conformally on $|z|<1$, then E corresponds to a set of measure zero on $|z|=1$, while if the exterior of C is so mapped, E corresponds to a set of positive measure. The result depends upon the following properties of the construction: (i) C lies in $\Re w\geqq 0$ and E is the common part of C and the real axis; (ii) E is of positive linear measure; (iii) no point of E is accessible from the interior of C by a rectifiable endcut. *W. K. Hayman* (Exeter).

Fox, E. N., and McNamee, J. The two-dimensional potential problem of seepage into a cofferdam. Philos. Mag. (7) **39**, 165–203 (1948).

Conformal mapping is used to obtain the flow lines and equipotential lines for fluid seepage into a cofferdam. The formal solution of Malhotra [Khosla, Bose and McKenzie-Taylor, Design of Weirs on Permeable Foundations, Central Board of Irrigation, India, Publ. no. 12 (1936)] is reproduced in detail. Included, in addition, are extensive numerical tables and curves giving the various quantities involved in terms of the physical constants of the cofferdam. The solution involves the Jacobian zeta function $Zn(u, k)$ and for this function it is proved that

$$Zn(u, k)=(\tfrac{1}{2}\pi/K')(1-u/K)+iZn(iu+K'-iK, k')$$

(using the standard notation for elliptic integrals) from which the authors deduce an identity relating the real and imaginary parts of $Zn(u, k)$. An infinite series is used to compute the real part of $Zn(K\psi_1+iK'\varphi, k)$ and tables are given for each increment of .1 in ψ_1 and φ, and for various values of k. Values for the imaginary part are easily obtained from the table by use of the authors' identity.

A. W. Goodman (New Brunswick, N. J.).

Piaggio, H. T. H., and Strain, M. N. The conformal transformation $Z=(lz^2+2mz+n)/(pz^2+2qz+r)$. J. London Math. Soc. **22** (1947), 165–167 (1948).

Elementary discussion of the transformation given in the title. *Z. Nehari* (St. Louis, Mo.).

Pfluger, Albert. Une propriété métrique de la représentation quasi conforme. C. R. Acad. Sci. Paris **226**, 623–625 (1948).

A quasi-conformal one-to-one mapping of $|z|<1$ onto itself is known to remain continuous and one-to-one on $|z|=1$. It is proved that any set on $|z|=1$ of inner measure zero is transformed into a set with the same property. Application: a quasi-conformal mapping of a plane region onto another preserves zero capacity of the boundary.

L. Ahlfors (Cambridge, Mass.).

Kufarev, P. P. A remark on integrals of Löwner's equation. Doklady Akad. Nauk SSSR (N.S.) **57**, 655–656 (1947). (Russian)

One form of Löwner's differential equation is

$$d(\log w)/dt=-(e^{i\alpha(t)}+w)/(e^{i\alpha(t)}-w),$$

where $\alpha(t)$ is a real continuous function of t, $t_0\leqq t\leqq T$ [Math. Ann. **89**, 103–121 (1923)]. Let $w=w(z, t)$, $t_0\leqq t\leqq T$, be the solution of this equation satisfying $w(z, t_0)=z$. The author gives an example of a function $\alpha(t)$ for which the corresponding solution $w(z, t)$ does not map $|z|<1$ onto $|w|<1$ minus a Jordan-arc slit. Taking $\alpha(t)=x^3$, $x=e^{-(t-t_0)}+i(1-e^{-2(t-t_0)})^{\frac{1}{2}}$, the corresponding solution is

$$w(z, t)=[x/(x^2+1)][z+x^2-\{(1-w_0)(x^4-z)\}^{\frac{1}{2}}],$$

$w(0, t)=0$. For $t>t_0$ this function maps $|z|<1$ onto $|w|<1$ minus the region bounded by the arc $\arg x\leqq\arg w\leqq\arg x^3$ of $|w|=1$ and the circular arc orthogonal to $|w|=1$ with end-points at $w=x$, $w=x^3$. *D. C. Spencer.*

Golusin, G. Method of variations in the theory of conform representation. II. Rec. Math. [Mat. Sbornik] N.S. **21(63)**, 83–117 (1947). (Russian. English summary)

In this paper [a sequel to part I, same Rec. N.S. **19(61)**, 203–236 (1946); these Rev. **8**, 325] the author applies variational methods to obtain results concerning extremal schlicht functions of the following three classes: (i) S, the class of functions $f(\zeta)=\zeta+c_2\zeta^2+\cdots$ regular and schlicht in $|\zeta|<1$; (ii) Σ, the class of functions $F(\zeta)=\zeta+a_0+a_1\zeta^{-1}+\cdots$ meromorphic and schlicht in $|\zeta|>1$; (iii) Σ_B, the class of functions $F(\zeta)$ which are meromorphic and schlicht in a given domain B of finite connectivity containing $\zeta=\infty$.

Given B and $F\in\Sigma_B$, let

$$J_F=\sum_{\nu,\nu'=1}^{n}\gamma_\nu\gamma_{\nu'}\log\{[F(\zeta_\nu)-F(\zeta_{\nu'})]/(\zeta_\nu-\zeta_{\nu'})\},$$

where the γ_ν are complex numbers and the ζ_ν $(\nu=1, \cdots, n)$ are given points of B. The author shows that the region of values of J_F as F ranges over Σ_B is a circle, thus generalizing a theorem of Grunsky which states that the region of $\log F'(\zeta_1)$ (ζ_1 a fixed point of B) is a circle. In particular, for functions F of Σ we obtain the inequality

$$|\log[F(\zeta_1)-F(\zeta_2)]/(\zeta_1-\zeta_2)|\leqq-\tfrac{1}{2}\log[(1-|\zeta_1|^{-2})(1-|\zeta_2|^{-2})].$$

Next, let P_F denote the upper bound of the quantity $|\prod_{\nu\neq\nu'}(a_\nu-a_{\nu'})|$ for all possible systems of n points a_ν, $\nu=1, \cdots, n$, on the boundary of the image of $|\zeta|>1$ by a function $F(\zeta)$ of Σ. It is shown that the upper bound P of P_F for F in Σ is attained for a function F of Σ which satisfies the differential equation

$$(dF/d\zeta)^2\sum_{\nu\neq\nu'}(F-a_\nu)^{-1}(F-a_{\nu'})^{-1}=n(n-1).$$

In the particular case $n=3$ the author finds the precise inequality $|(a_1-a_2)(a_1-a_3)(a_2-a_3)|\leqq12\sqrt{3}$. It is pointed out as a corollary that, if c_1, c_2, c_3 are any three points on three rays issuing symmetrically from the origin and lying in the boundary of the image of $|\zeta|<1$ by $f(\zeta)$ of S, then $|c_1c_2c_3|\geqq\frac{1}{4}$. *D. C. Spencer* (Stanford University, Calif.).

Golusin, G. Method of variations in the theory of conform representation. III. Rec. Math. [Mat. Sbornik] N.S. **21(63)**, 119–132 (1947). (Russian. English summary)

[Cf. the preceding review.] Let S be the class of functions $f(\zeta)=\zeta+c_2\zeta^2+c_3\zeta^3+\cdots$ which are regular and schlicht in $|\zeta|<1$. Given m constants $c^0_{n_k}$, $k=1, \cdots, m$, the author considers the subclass $S(c^0_{n_1}, \cdots, c^0_{n_m})$ of S for which $c_{n_k}=c^0_{n_k}$, $k=1, \cdots, m$, and finds a differential equation for functions of $S(c^0_{n_1}, \cdots, c^0_{n_m})$ which maximize $|c_n|$, $n\neq n_k$, $k=1, \cdots, m$. A differential equation is also found for functions which maximize the quantity

$$J_f=\min\{|f(\zeta_1)|, \cdots, |f(\zeta_k)|\}$$

with respect to the whole class S.

D. C. Spencer (Stanford University, Calif.).

Goodman, A. W. On some determinants related to p-valent functions. Trans. Amer. Math. Soc. **63**, 175–192 (1948).

The paper centres on the family of functions $w=\sum_0^\infty b_nz^n=S(z)=P(u)/Q(u)$, $u=z/(1-z)^2$, where $P(u)$ and $Q(u)$ are polynomials of degree p at most. The roots u_k of $Q(u)$ are restricted to satisfy $u_k\leqq-\frac{1}{4}$ and $S(z)$ is then regular and at most p-valent for $|z|<1$. The coefficients $b_0, b_1, \cdots, b_p$ are arbitrary by choice of $P(u)$ and b_n is a linear function of $b_0, b_1, \cdots, b_p$. If $|b_k|\leqq B_k$ for $k=0, 1, \cdots, p$, then b_n is shown to attain its maximum modulus $\sum_0^pD_kB_k$ when (i) $Q(u)$ is constant and (ii) $b_k=(-)^kB_k$, $k=0, 1, \cdots, p$. The conjecture is therefore made that for all functions $f(z)=\sum_0^\infty a_nz^n$ regular and at most p-valent for $|z|<1$ the inequality $|a_n|\leqq\sum_0^pD_k|a_k|$ holds for $n>p$. The explicit

formulae for b_n in terms of $b_0, b_1, \cdots, b_p$ involve determinants of the form $\|\sin j\theta_i\|$ except that the row $j=k$ is replaced by $j=n$, and the theorem stated depends on inequalities for these determinants and explicit calculations for analogous determinants. An elementary proof of a related classical inequality ascribed to O. H. Mitchell [Amer. J. Math. 4, 341–344 (1882)] is added. *A. J. Macintyre.*

Erim, Kerim. Über beschränkte Funktionen, die in vorgeschriebenen n Punkten gleiche Werte annehmen. Rev. Fac. Sci. Univ. Istanbul (A) 12, 237–254 (1947). (German. Turkish summary)

The author obtains the least upper bound for $f'(0)$, when $f(z)$ is regular in $|z|<R$, $|f(z)|<M$, $f(0)=0$, $f(z_1)=\cdots=f(z_n)$. The case $n=2$ was previously treated by Carathéodory [Monatsh. Math. Phys. 43, 225–241 (1936)] and his method extends to the general case.

W. K. Hayman (Exeter).

Schubart, Hans. Ganzwertigkeit und verallgemeinerte Dirichletsche Reihen. Math. Ann. 120, 202–216 (1948).

Let $g(s)=\sum_1^\infty a_n \exp(-\lambda_n s)$, absolutely convergent for $\Re(s)\geqq 0$, where λ_n is a complex number, $|\arg(\lambda_n)|\leqq c<\pi/2$ for every n, and such that the numbers $e^{-\lambda_n}$ form a set of isolated points. Let $g(n)$ be an integer for $n=n_0, n_0+1, \cdots$. Then, for all $n\geqq n_0$, $g(n)=\sum_1^m A_k(e^{\gamma_k})^n$, where A_k is an algebraic number, and e^{γ_k} an algebraic integer. The author applies this to investigate conditions under which $g(s)=(u^s-1)/(v^s-1)$ is integral valued, or almost integral valued, by which is meant that $g(n)=b_n+\epsilon_n$, where b_n is an integer, and $|\epsilon_n|\leqq\theta^n$, where $0\leqq\theta<1$. [Publication of this was delayed; for more recent results, see C. Pisot, C. R. Acad. Sci. Paris 222, 988–990, 1027–1028 (1946); 224, 438–440 (1947); these Rev. 8, 23, 454.] *R. C. Buck.*

Čakalov, Lyubomir N. On a representation of entire functions of degree zero. Doklady Akad. Nauk SSSR (N.S.) 58, 535–538 (1947). (Russian)

The author proves that a necessary and sufficient condition for an entire function to satisfy $\lim_{r\to\infty} r^{-1}M(r)=0$ is that it should be representable in the form

$$\sum_0^\infty A_n z(z-1)\cdots(z-n+1)/n!,$$

with $\lim|A_n|^{1/n}=0$. [The theorem is known; see, e.g., J. M. Whittaker, Interpolatory Function Theory, Cambridge University Press, 1935, pp. 52–53.] As an application, the author proves Wigert's theorem giving a necessary and sufficient condition for a function to be an entire function of $(1-z)^{-1}$. [For a similar proof cf. Whittaker, op. cit., p. 56.] *R. P. Boas, Jr.* (Providence, R. I.).

Al'per, S. Ya. On the overconvergence of series of polynomials. Doklady Akad. Nauk SSSR (N.S.) 59, 625–627 (1948). (Russian)

Let C be a Jordan curve bounding a simply-connected region and let the function $w=\varphi(z)=z+\alpha_0+\alpha_1/z+\alpha_2/z^2+\cdots$ provide a one-to-one conformal mapping of the region exterior to C onto the region $|w|>r$. For each R $(R>r)$, let C_R denote the curve in the z-plane which is mapped into the circle $|w|=R$. Let $\{p_n(z)\}$ denote a sequence of polynomials of degree n $(n=0, 1, \cdots)$ with the property that for each positive ϵ and each R there exists a constant M such that the inequality $|p_n(z)|<M(R+\epsilon)^n$ holds on C_R. If $\{a_n\}$ is a sequence of constants satisfying the condition $\limsup|a_n|^{1/n}=1/\rho$ $(\rho>r)$, the series (1) $\sum a_n p_n(z)$ converges to a holomorphic function $f(z)$ in the interior of C_ρ. The following result is announced. If the sequence $\{a_n\}$ has Ostrowski gaps and the direct analytic continuation of $f(z)$ is regular at the point z_0 on C_ρ, the subsequence of those partial sums of (1) that correspond to the gaps converges to $f(z)$ uniformly in some neighborhood of z_0. The author states that the proof utilizes mapping theory and variations of the method of Ostrowski that yields the theorem for the special case of Taylor series. He applies his result to series of Faber polynomials, to series of polynomials orthogonal on C and to Newton and Abel series. In the special case where the Ostrowski gaps are Hadamard gaps, the curve C_ρ is a natural boundary for the function $f(z)$.

G. Piranian (Ann Arbor, Mich.).

Arpiarian, Noubar. Sur les suites de fonctions orthogonales par rapport à un ensemble de courbes ou de domaines différents. C. R. Acad. Sci. Paris 226, 771–772 (1948).

Arpiarian, Noubar. Orthogonalité sur un domaine et sur son contour. Relations entre intégrales de lignes et intégrales de surface. C. R. Acad. Sci. Paris 226, 865–866 (1948).

The author discusses the relation between the orthogonalities of a family of functions $f_n(z)$ of a complex variable z over a simply connected domain D and on its contour C. The nonweighted orthogonalities are independent properties, but the author shows that the weight-functions $d(z)$, for the domain D, and $c(z)$, for the curve C, related by $d(z)=c(z)|h'(z)|$, where the single-valued function $h(z)=Z$ gives the conformal representation of D on the circle $Z\leqq 1$, ensure one of the two orthogonalities when the other is given, provided the following conditions are satisfied. Using the "isograms" C_t to the curve C and D_t to the domain D, defined by $|Z|\leqq t$, for $0<t\leqq 1$, the author proves that one orthogonality entails the other if the postulated orthogonality (over $D=D_1$ or on $C=C_1$) extends to the family of all isograms (D_t or C_t for $0<t<1$). In particular, for each family of isograms D_t and C_t there is at least one family $f_n(z)$ orthogonal over D and C and such that $d(z)=c^2(z)$ for it. It is $f_n(z)=Z^n=[h(z)]^n$ since in this case $c(z)=|h'(z)|$. A necessary and sufficient condition for orthogonality on C, if $f_n(z)$ is orthogonal with the weight-function $d(z)$ over D, as found by the author, is $I(t)=o(|t-1|)$ for $t\to 1$, where $I(t)$ denotes the double integral of the product $f_m(z)\bar{f}_n(z)d(z)$, $n\neq m$, over the isogram domain D_t ($t<1$ or $t>1$), the weight-function $c(z)$ being equal to $d(z)|h'(z)|$.

E. G. Kogbetliantz (New York, N. Y.).

Doss, Shafik. Sur le comportement asymptotique des zéros de certaines fonctions d'approximation. Ann. Sci. École Norm. Sup. (3) 64 (1947), 139–178 (1948).

The present investigation was carried out under the guidance of M. Fekete and a considerable part of the results contained in this paper are due to him. A sequence $\{p_n(z)\}$ of polynomials is considered which converges uniformly to an analytic function $f(z)\not\equiv 0$ in a certain region R. Let z_0 be a k-tuple zero of $f(z)$ in R; by Hurwitz's theorem $p_n(z)$ will have exactly k zeros z_{ni}, $1\leqq i\leqq k$, in a sufficiently small neighborhood of z_0 provided n is sufficiently large. The purpose is to study the behavior of the differences $z_{ni}-z_0$ for large n. The following results may be mentioned. (1) Let $p_n(z)$ be the nth partial sum of the power series of $f(z)$ with a radius of convergence ρ, $0<\rho<\infty$, and let $z_0\neq 0$.

Then lim sup $|z_{n,i}-z_0|^{k/n}=|z_0|/\rho$. (2) If $\rho=\infty$ and $f(z)$ is of finite order λ, the same result holds on replacing k/n by $k/n\log n$ and $|z_0|/\rho$ by $\exp(-1/\lambda)$. (3) Let $p_n(z)$ be the nth Cesàro mean of integral order r, $0<\rho\leqq\infty$, $r\geqq 1$. Then for $r\geqq k$ we have lim $n(1-z_0/z_{n,i})=\alpha_i$, where α_i are the zeros of a certain polynomial of degree k depending on k and r. Similar results hold for $r<k$. (4) Let $p_n(z)$ be the nth Jensen polynomial, $\rho>0$. Then lim $n^1(z_{n,i}/z_0-1)=\alpha_i$ exists and coincides with the roots of the Hermite polynomial of degree k. (5) A further sequence is studied which was considered earlier by Fekete and defined as follows:

$$p_n(z)=z^{n+r+1}(d/dz)^r\{z^{-n-1}f_n(z)\},$$

where $f_n(z)$ is the nth partial sum. (6) The Hölder means of the partial sums are studied. (7) Sequences of "maximal convergence" in the sense of Walsh are investigated in a certain domain from the same point of view.

G. Szegö (Stanford University, Calif.).

Baganas, Nicolas. Sur un problème de M. P. Montel et les intégrales pseudo-abéliennes. C. R. Acad. Sci. Paris **226**, 988–990 (1948).

Announcement of further results [cf. the same vol., 545–546 (1948); these Rev. **9**, 343]. In the present note the author introduces the notion of pseudo-Abelian integrals, i.e., Abelian integrals representable as a sum of logarithms of algebroid or algebraic functions, and relates this notion to the problems of exceptional functions with which he is concerned. *M. Heins* (Providence, R. I.).

Bochner, S. On compact complex manifolds. J. Indian Math. Soc. (N.S.) **11**, 1–21 (1947).

Il est bien connu que la surface de Riemann d'une courbe algébrique de genre $p>1$ n'a qu'un nombre fini d'automorphismes (préservant la structure analytique complexe). L'auteur envisage d'étendre ce théorème à certaines variétés compactes de dimension (complexe) quelconque; il s'agit uniquement de variétés munies d'une structure analytique complexe. A cette occasion, il passe en revue divers problèmes ou résultats, anciens ou nouveaux, de la théorie des fonctions analytiques de plusieurs variables complexes.

D'abord le groupe Γ de tous les automorphismes (analytiques complexes) de S est localement compact pour la topologie de la convergence uniforme; donc la composante connexe de l'identité dans Γ est un groupe de Lie [résultat établi par H. Cartan [Sur les groupes de transformations analytiques, Actual. Sci. Indus., no. 198, Hermann, Paris, 1935] étendu récemment par Bochner et Montgomery [Ann. of Math. (2) **47**, 639–653 (1946); ces Rev. **8**, 253] aux variétés deux fois différentiables]. Deux cas sont possibles: dans le premier, Γ est discret; dans le second, S admet une transformation infinitésimale, ou, ce qui est équivalent, il existe sur S un champ de vecteurs analytiques contravariant. Pour prouver (si possible) que le groupe Γ d'une variété S est fini, on prouvera: (1) l'inexistence d'un champ contravariant; (2) que Γ est non seulement discret, mais fini.

L'auteur donne des conditions suffisantes pour que ces deux conditions soient remplies. Par exemple, il suffit que le domaine de recouvrement universel $\tilde{S}$ de S se représente analytiquement sur un domaine borné de l'espace de k variables complexes, et qu'une condition supplémentaire soit vérifiée (qui, en gros, est relative au comportement du groupe de tous les automorphismes de $\tilde{S}$ lorsque le transformé d'un point fixe tend vers la frontière de $\tilde{S}$). Cette condition est remplie si S est une courbe algébrique de genre $p>1$ ($\tilde{S}$ étant le cercle-unité); plus généralement, si $\tilde{S}$ se représente sur une boule ou un polycylindre à k dimensions complexes.

L'inexistence d'un champ de tenseurs analytiques contravariant peut aussi être prouvée en supposant qu'il existe sur S une "métrique kählerienne" [au sens de A. Weil, Comment. Math. Helv. **20**, 110–116 (1947); ces Rev. **9**, 65] pour laquelle la courbure de Ricci est non positive et non identiquement 0. S'il existe une métrique kählerienne pour laquelle la courbure de Ricci est non négative et non identiquement 0, il n'y a pas de tenseur analytique covariant.

L'auteur évoque aussi certains espaces de matrices. Il émet plusieurs conjectures intéressantes [voir notamment p. 6, et "théorème 7"]. *H. Cartan* (Paris).

Theory of Series

Jarden, Dov. A bibliography of the Fibonacci sequence. Riveon Lematematika **2**, 36–45 (1948). (Hebrew)

The entries (over 250) in the bibliography are in the original languages.

Jarden, Dov. Generalization of Landau's theorem on the series of the inverses of Fibonacci numbers with even subscripts. Riveon Lematematika **2**, 34 (1948). (Hebrew)

Let $L(x)$ denote Lambert's function $\sum_{n=1}^{\infty}x^n(1-x^n)^{-1}$ and define the second order recurring series u_n by $u_0=0$, $u_1=1$, $u_{n+1}=Pu_n+u_{n-1}$. Finally let α be the root of $x^2-Px-1=0$ which lies inside the unit circle. Then

$$\sum_{n=1}^{\infty}1/u_{2n}=(P^2+4)^{\frac{1}{2}}\{L(\alpha^2)-L(\alpha^4)\}.$$

For $P=1$ we have the Fibonacci sequence and the result is a theorem of Landau [Bull. Soc. Math. France **27**, 298–300 (1899)]. *D. H. Lehmer* (Berkeley, Calif.).

Orts, J. M.ª Differential properties of recurrent series. Revista Mat. Hisp.-Amer. (4) **7**, 109–116 (1947). (Spanish)

It is pointed out that if the coefficients in the series

(1) $$y=\sum a_nx^n/n!$$

satisfy a linear recursion relation of the form

(2) $$\sum_{j=0}^{q}\mu_ja_{r+j}=0, \qquad r=0, 1, 2, \cdots,$$

where the μ_j are constants, then y is given by an expression of the form

$$y=\sum_{j=0}^{q}c_je^{\alpha_jx},$$

so that the differential equation

$$\sum_{j=0}^{q}\mu_jy^{(r+j)}=0, \qquad r=0, 1, 2, \cdots,$$

which, by (1) and (2), clearly is satisfied at the origin for $r=0$, actually is satisfied identically. The author now considers series of the form $y=\sum'a_nx^n$, for which the recursion relation (2) is satisfied, points out that any solution y must be a rational function of x, and determines the corresponding differential equation. Several special cases are treated.

E. F. Beckenbach (Los Angeles, Calif.).

Duffin, R. J. A generalization of the ratio test for series. Amer. Math. Monthly **55**, 153–155 (1948).

The following two theorems are proved. (1) A series $a_1+a_2+\cdots$ of real numbers is convergent if $a_n\to 0$ and if the ratios a_{n+1}/a_n satisfy the inequality $-1\leqq a_{n+1}/a_n<r<1$, where $0<r<1$. (2) A series $a_1+a_2+\cdots$ of complex numbers is convergent if $a_n\to 0$ and if the ratios a_{n+1}/a_n lie in or on a polygon which is contained in the closed unit circle and touches the circle at one and only one point $w\neq 1$. There are two additional theorems. *T. Fort* (Athens, Ga.).

Biernacki, Mieczysław. Sur une propriété des suites à termes positifs. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 1, 19–21 (1946). (French. Polish summary)

Proof of a theorem equivalent to the following: if p_n is a sequence of positive numbers such that

$$\lim_{n\to\infty} p_n/(p_1+p_2+\cdots+p_n)=0,$$

then

$$\liminf_{n\to\infty} p_n/(p_1b_n+p_2b_{n-1}+\cdots+p_nb_1)=0$$

whenever $\sum b_n$ is a divergent series of positive terms. *R. P. Agnew* (Ithaca, N. Y.).

Ogieveckiĭ, I. E. A generalization of the theorem of Dirichlet and Hadamard for quasi-uniformly convergent series. Doklady Akad. Nauk SSSR (N.S.) **56**, 791–794 (1947). (Russian)

Two classic theorems on convergence of series of the type $\sum a_nu_n$ are modified to cover quasi-uniform convergence of series of functions. A series $\sum f_k(x)$ is said to converge quasi-uniformly over a set E if it converges over E and to each pair (ϵ, N) of positive numbers corresponds a number $N'>N$ such that for each fixed x in E the inequality $|\sum_{k=n}^{n'} f_k(x)|<\epsilon$ holds for at least one integer $n=n(x)$ in the interval $N<n<N'$. (1) If $a_n(x)$ converges to 0 uniformly over E then the series $\sum a_n(x)u_n(x)$ converges quasi-uniformly over E, whenever $\sum u_n(x)$ has partial sums uniformly bounded over E, if and only if the series (*) $\sum_{n=1}^{\infty}|a_n(x)-a_{n+1}(x)|$ converges quasi-uniformly over E. (2) If the functions $a_n(x)$ are uniformly bounded over E, then the series $\sum a_n(x)u_n(x)$ converges quasi-uniformly over E, whenever $\sum u_n(x)$ converges uniformly over E, if and only if the series (*) converges quasi-uniformly over E. The proofs are easy applications of the Abel partial summation formula. *R. P. Agnew* (Ithaca, N. Y.).

Ogieveckiĭ, I. E. On criteria of convergence for a class of double series. Doklady Akad. Nauk SSSR (N.S.) **58**, 1893–1896 (1947). (Russian)

If $\alpha_{ij}(x)$ is a double sequence of functions such that

$$\sum_{i,j=1}^{\infty}|\Delta_{ij}\alpha_{ij}(x)|<K(x),$$

$\lim_{n\to\infty}\Delta_{i0}\alpha_{in}(x)=0$, $\lim_{m\to\infty}\Delta_{0j}\alpha_{mj}=0$ and $\lim_{m,n\to\infty}\alpha_{mn}=0$, where $\Delta_{ij}\alpha_{ij}=\alpha_{ij}-\alpha_{i,j+1}-\alpha_{i+1,j}+\alpha_{i+1,j+1}$, $\Delta_{i0}\alpha_{ij}=\alpha_{ij}-\alpha_{i+1,j}$ and $\Delta_{0j}\alpha_{ij}=\alpha_{ij}-\alpha_{i,j+1}$, then the double series $\sum\alpha_{ij}(x)u_{ij}(x)$ converges whenever $\sum u_{ij}(x)$ has bounded partial sums. For cases in which the differences are nonnegative, the hypotheses are simplified and are strengthened (by requiring uniformity of limits) to give uniform convergence of $\sum\alpha_{ij}(x)u_{ij}(x)$ whenever $\sum u_{ij}(x)$ has uniformly bounded partial sums. *R. P. Agnew* (Ithaca, N. Y.).

Nicolesco, Miron. Sur le théorème fondamental de convergence de Pringsheim pour les suites doubles. Bull. Math. Soc. Roumaine Sci. **47**, 3–9 (1946).

Proofs of familiar Pringsheim theorems about the existence of $\lim_{m,n\to\infty} s_{mn}$ and such iterated limits as the limit as $n\to\infty$ of $\limsup_{m\to\infty} s_{mn}$. *R. P. Agnew* (Ithaca, N. Y.).

Borgers, Alfons. Contribution to the arithmetical theory of Cesàro's method of summability. Verh. Vlaamsche Akad. Kl. Wetensch. **8**, no. 19, 207 pp. (1946). (Dutch. English summary)

This thesis is a survey of parts of the theory of the Cesàro methods C_r of summability of complex orders r. Many familiar results of the theory are established, and in some cases improved, sometimes by use of the known general theory of sequence-to-sequence transformations

$$\sigma_n=\sum_{k=0}^{n}a_{nk}s_k, \qquad n=0, 1, 2, \cdots, \tag{1}$$

which the author gives with references, but without proofs, in the first chapter. One of the theorems, G_3 of page 26, involving (1), should have the hypothesis $\lim_{p\to\infty} a_{n+p,k+p}=0$, $n, k=0, 1, 2, \cdots$, instead of the hypothesis that the limits exist. In connection with the Silverman-Toeplitz theorem, the reference should be Silverman's thesis [On the definition of the sum of a divergent series, University of Missouri Studies 1, no. 1, accepted 1910, published 1913], not the later paper by Silverman [Trans. Amer. Math. Soc. **17**, 284–294 (1916)] which the author cites. The latter part of chapter 1 gives facts and formulas involving factorials and binomial coefficients and proves Mercer's theorem [Proc. London Math. Soc. (2) **5**, 206–224 (1907)] and several related theorems.

In chapter 2, the author introduces the Cesàro methods C_r of complex orders r, confining nearly all his attention to orders for which $\Re(r)>-1$. His assertion that previous authors confined themselves to real orders is surprising since many authors have treated complex orders. He treats relations among the methods C_r, the effect of removing initial elements from a sequence, Cauchy products, summability of $\sum u_n$ when its subseries $\sum_{j=0}^{\infty}u_{jk+m}$ $(m=1, 2, \cdots, k)$ are all summable, and C_β summability of C_α transforms of given series. The Andersen theorem [if $r>-1$, if $\sum u_n$ is bounded C_r, and if $\sum u_n$ is summable C_s for at least one s for which $\Re(s)>r$, then $\sum u_n$ is summable C_s for every s for which $\Re(s)>r$] and related important theorems are given well-deserved prominence.

Chapter 3 gives familiar necessary conditions for summability and relates the methods C_r to the Hölder methods H_r and the Euler-Abel power series method. Chapter 4 gives and improves several known theorems about summability of $\sum a_nb_n$. Chapter 5 gives, and extends to nonintegral orders, known theorems of the nature of the following. A series $\sum u_n$ with partial sums s_n is summable C_1 to s if and only if

$$\lim_{n\to\infty}\left\{s_n+(n+1)\sum_{k=n+1}^{\infty}u_k/k\right\}=s.$$

Chapter 6 gives some of the many Tauberian theorems, for the Cesàro and Euler-Abel methods, which have been proved in the last 50 years. Methods that have been devised for proofs of some of these theorems are used to prove and extend others. The final chapter, chapter 7, gives expositions of absolute Cesàro summability and strong Cesàro summability, the author's main claim for novelty being the follow-

ing theorem. The fact that $\sum u_n$ is absolutely summable C_r does not imply existence of an order s with $\Re(s)<\Re(r)$ such that $\sum u_n$ is summable C_s.

With each chapter there is associated a list of references, the number of different books and papers cited being about 125. Each list contains many references which, except for a few typographical errors, are precise and accurate. It is inevitable that, because so much work has been done with the Cesàro methods C_r, many important results and papers are unmentioned, and that some sources claimed to be original sources are in fact not original sources. Nevertheless many who work with Cesàro summability and its applications will find this thesis worthy of study and of use as a source of ideas, theorems, formulas and references.

R. P. Agnew (Ithaca, N. Y.).

Rajagopal, C. T. Cesàro summability of a class of functions. J. Indian Math. Soc. (N.S.) **11**, 22–27 (1947).

The author states that, if $f(x)$ is periodic, with zero integral over a period, and if $a(x)=x^{k-1}$, then (i) $f(x)a(x)=o(1)$ (C,k) as $x\to\infty$ and (ii) $\int_0^x f(x)a(x)dx$ is summable (C,k) where k is a positive integer. A series analogue of the second result has been given by Macphail [Bull. Amer. Math. Soc. **47**, 483–487 (1941); these Rev. **3**, 148].

The results are correct but the author's proof is fallacious. He assumes that the solution of the difference equation $\Delta_\omega^k S_k(x)=0$, where $\Delta_\omega\varphi(x)=\varphi(x+\omega)-\varphi(x)$, is

$$S_k(x)=\sum_1^{k-1} c_r x^r$$

plus a periodic function ($S_k(x)$ is the kth integral of $f(x)a(x)$ with base $x=0$; ω is the period). This is false if $k>1$, for instance because by $k-1$ differentiations it would follow that $S_1(x)$ is a periodic function, whereas if $f(x)=\cos x$ and $a(x)=x$, it is found that $S_1(x)=x\sin x+\cos x-1$. On the other hand, since $S_k(x)$ is continuous, the difference equation does imply $S_k(x)=O(x^{k-1})=o(x^k)$, which is the first result, and the second is deduced after integration by parts as before. In his statement that the function $a(x)=x^{k-1}$ may be replaced by a more general function the author's condition (7) is insufficient to deal with the terms under (6). On the other hand, the conditions stated in the footnote are sufficient for the more general result. [In the general case, the difference equation will not hold, but may be avoided by an appeal to induction after a preliminary integration by parts.] *L. S. Bosanquet* (London).

Rajagopal, C. T. Some limit theorems. Amer. J. Math. **70**, 157–166 (1948).

Let $p_n>0$, $q_n>0$, $\sum p_n=\infty$, $\sum q_n=\infty$,

$$t_n(p)=\sum_1^n p_r s_r/\sum_1^n p_r, \quad t_n(q)=\sum_1^n q_r s_r/\sum_1^n q_r.$$

It is known that the existence of $\lim t_n(p)$ implies that of $\lim t_n(q)$ if either (a) q_n/p_n decreases or (b) q_n/p_n increases so that $q_n/\sum_1^n q_r<kp_n/\sum_1^n p_r$ for some $k>1$ [Cesàro, Atti Accad. Lincei. Rend. (4) **4**$_1$, 452–457 (1888); Hardy, Quart. J. Math. **38**, 269–288 (1907); Bromwich, An Introduction to the Theory of Infinite Series, 2d ed., Macmillan, London, 1926, p. 427]. Thus the result holds if $p_n=n^\alpha$, $q_n=n^\beta$, $\alpha>-1$, $\beta\geqq-1$, but not if $\alpha=-1$, $\beta>-1$, as shown by the example $s_n=n^i$. Here the author shows that if $\bar\lambda$, $\underline\lambda$, $\bar l$, $\underline l$ denote the superior and inferior limits of $t_n(p)$ and $t_n(q)$, then in case (a) $\underline\lambda\leqq\underline l\leqq\bar l\leqq\bar\lambda$, while in case (b) $\bar\lambda-k(\bar\lambda-\underline\lambda)\leqq\underline l\leqq\bar l\leqq\underline\lambda+k(\bar\lambda-\underline\lambda)$. For example, if $p_n=n^\alpha$, $q_n=n^\beta$, $\beta>\alpha>-1$, then $k=(\beta+1)/(\alpha+1)+\epsilon$. Applications include oscillation theorems of Fuchs, Karamata, Knopp and Rademacher. *L. S. Bosanquet* (London).

Lee, S. C. A note on trigonometrical series. J. London Math. Soc. **22** (1947), 216–219 (1948).

The author investigates the Riemann method of summation of a series $\sum a_n$; he shows that (i) $\sum_1^n a_r\to+\infty$ and (ii) $-n^{-1}<a_n<o(1)$ imply $\lim_{h\to0}\sum_1^\infty a_n\sin^2 nh/n^2h^2=\infty$; and that (ii) cannot be replaced by any condition of the form $|a_n|\leqq W_n/n$, $W_n\to+\infty$. *G. Lorentz* (Tübingen).

Lorentz, G. G. Tauberian theorems and Tauberian conditions. Trans. Amer. Math. Soc. **63**, 226–234 (1948).

The author proves that the condition that for each $\epsilon>0$ there exists a sequence of positive numbers n_k for which $n_{k+1}/n_k\geqq q>1$ and (1) $\sum_{n_k<n<n_{k+1}} c_n<\epsilon$ is necessary and sufficient for (2) $u_n=O(c_n)$ to be a Tauberian condition for Abel or Cesàro summability of the series $\sum u_n$. If (2) is replaced by the one-sided condition $u_n\leqq Kc_n$ for some positive constant K, the theorem remains true if (1) is replaced by the stronger condition $\sum_{n_k\leqq n<n_{k+1}} c_n<\epsilon$. *H. R. Pitt.*

Delange, Hubert. Théorèmes taubériens pour les séries doubles. C. R. Acad. Sci. Paris **225**, 855–856 (1947).

The author considers the double series $\sum\sum a_{mn}\xi^{-mx-ny}$, converging absolutely for $x>0$, $y>0$ to $S(x,y)$; and shows how the limits of partial sums of $\sum\sum a_{mn}$ are related to the limits of $S(x,y)$ as $x\to0$, $y\to0$ when a_{mn} satisfy appropriate Tauberian conditions. *H. R. Pitt* (Belfast).

Delange, Hubert. Théorèmes taubériens pour les séries multiples de Dirichlet. C. R. Acad. Sci. Paris **226**, 377–379 (1948).

Results analogous to those of the preceding note are given for the more general series $\sum\sum a_{m,n}e^{-\lambda_m z-\mu_n z'}$, where $\{\lambda_n\}$, $\{\mu_n\}$ are real sequences and z, z' are complex.

H. R. Pitt (Belfast).

Verblunsky, S. A note on Tauberian theorems and harmonic functions. J. London Math. Soc. **22** (1947), 210–216 (1948).

According to an erratum slip, the author wishes to withdraw the paper, since it contains a mistake.

Fourier Series and Generalizations, Integral Transforms

Salem, R., and Zygmund, A. On lacunary trigonometric series. II. Proc. Nat. Acad. Sci. U. S. A. **34**, 54–62 (1948).

Les auteurs complétent une note précédente [mêmes Proc. **33**, 333–338 (1947); ces Rev. **9**, 181] sur les séries trigonométriques lacunaires du type $\sum_{k=1}^\infty(a_k\cos n_kx+b_k\sin n_kx)$ où $n_{k+1}/n_k>q>1$, et démontrent en particulier que: (a) les propositions (i) à (vi) de cette note restent valables si les n_k ne sont plus assujettis à être entiers (même principe de démonstration, convenablement amélioré); (b) en supposant les n_k entiers, en considérant les "moyennes" $\sigma_m(x)=\sum_{k=1}^\infty\alpha_{mk}(a_k\cos n_kx+b_k\sin n_kx)$ où l'indice m tend vers $+\infty$, avec $c_k=(a_k^2+b_k^2)^{\frac12}$, $C_n=2^{-\frac12}(\sum_{k=1}^n c_k^2)^{\frac12}$, $C_m^*=2^{-\frac12}(\sum_{k=1}^\infty\alpha_{mk}^2c_k^2)^{\frac12}$, si $C_m^*<\infty$ quel que soit m, si $\lim_{m\to\infty}\alpha_{mk}=1$ quel que soit k, si $\lim_{n\to\infty}C_n=+\infty$ et si

$\max_{1\leq k<n} |c_k a_{mk}|/C_m^* \to 0$ lorsque $m\to\infty$, la fonction de distribution de $\sigma_m(x)/C_m^*$ tend vers la loi de Laplace d'espérance mathématique nulle et d'écart moyen quadratique 1, lorsque $m\to\infty$; (c) en posant

$$S_n(x)=\sum_{k=1}^{n}(a_k\cos n_k x+b_k\sin n_k x)$$

où on suppose, non pas que les n_k satisfont à $n_{k+1}/n_k>q>1$, mais forment un ensemble de nombres linéairement indépendants, si $\lim_{n\to\infty} C_n=+\infty$ et $c_k=o(C_k)$, la fonction de distribution de $S_n(x)/C_n$ tend vers la loi de Laplace lorsque $n\to\infty$. *R. Fortet* (Caen).

Kac, M., Salem, R., and Zygmund, A. A gap theorem. Trans. Amer. Math. Soc. 63, 235–243 (1948).

Soient une suite $\{\lambda_k\}$ de nombres positifs tels que $\lambda_{k+1}/\lambda_k>q>1$, ou q est un nombre indépendant de k, et une fonction réelle $f(x)$ d'une variable réelle x, périodique de période 2π, de la classe L^2, telle que $\int_0^{2\pi} f(x)dx=0$; en désignant par $s_n(x)$ la somme des n premiers termes de la série de Fourier de $f(x)$ et en supposant qu'il existe un nombre positif σ tel que $\int_0^{2\pi}(f-s_n)^2dx=O(\log n)^{-\sigma}$, les auteurs demontrent que: (a) si $\sigma>2$, pour toute suite de constantes réelles c_k telles que $\sum_k c_k^2\log^2 k<\infty$, la série $\sum_k c_k f(\lambda_k x)$ converge presque-partout; (b) si $\sigma\leqq 2$, la série $\sum_k f(\lambda_k x)/k^{1-\delta}$, ou δ est une constante indépendante de k, converge presque-partout pourvu que $\delta<\sigma/4$; il en résulte que

$$\lim_{m\to\infty}\{f(x)+\sum_{k=1}^{m} f(\lambda_k x)\}/(m+1)=0.$$

La méthode des auteurs leur permet également de démontrer que: étant donnée une variable réelle t sur l'intervalle $(T_0,+\infty)$ et une fonction aléatoire réelle $X(t)$ de t définie sur $(T_0,+\infty)$, si $E[X(t)]=0$ et $E[X(t)^2]=1$ et s'il existe un nombre positif α tel que $|E[X(t)X(t+u)]|<|u|^{-\alpha}$ au moins pour $|u|$ assez grand, on a: $\lim_{T\to+\infty} T^{-1}\int_{T_0}^{T_0+T} X(t)dt=0$ presque-surement [il s'agit de l'intégrale aléatoire de $X(t)$ en moyenne quadratique]. *R. Fortet* (Caen).

Men'šov, D. On the convergence in measure of trigonometric series. Doklady Akad. Nauk SSSR (N.S.) 59, 849–852 (1948). (Russian)

If $f(x)$ is measurable and finite almost everywhere on $(-\pi,\pi)$, the author has shown how to construct a trigonometric series converging almost everywhere to $f(x)$ [Rec. Math. [Mat. Sbornik] N.S. 9(51), 667–692 (1941); these Rev. 3, 106]. He now asks whether the restriction of finiteness can be dropped, and gives a partial answer in which convergence is replaced by convergence in measure. He says that $f_n(x)\to f(x)$ in measure (where $f(x)$ is not necessarily finite almost everywhere but the $f_n(x)$ are) if $f_n(x)=g_n(x)+\alpha_n(x)$, g_n and α_n are finite almost everywhere, $g_n(x)\to f(x)$ almost everywhere and $\alpha_n(x)\to 0$ in measure. Then for every measurable $f(x)$, defined but not necessarily finite almost everywhere, there is a trigonometric series, with coefficients tending to 0, whose partial sums converge to $f(x)$ in measure.

Two more general theorems are stated. A function $F(x)$, measurable but not necessarily finite almost everywhere, is called the upper limit in measure of the sequence $\{f_n(x)\}$ if $\lim \text{meas } E[f_n(x)>\varphi(x)]=0$ for every $\varphi(x)$ such that $\varphi(x)>F(x)$ if $F(x)<+\infty$, $\varphi(x)=F(x)$ if $F(x)=+\infty$, while $\limsup \text{meas } E[f_n(x)>\psi(x)]\cdot E[F(x)>\psi(x)]>0$ for every $\psi(x)$ such that $\text{meas } E[F(x)>\psi(x)]>0$. Then, given two measurable functions $F(x)$ and $G(x)$, with $G(x)\leqq F(x)$ almost everywhere on $(-\pi,\pi)$, there is a trigonometric series, with coefficients tending to zero, such that $F(x)$ and $G(x)$ are the upper and lower limits in measure of its partial sums, and having either of the following properties: (a) for every $\psi(x)$ such that $G(x)\leqq\psi(x)\leqq F(x)$, a sequence of the partial sums of the series tends to $\psi(x)$ almost everywhere; (b) if $\psi_1(x),\cdots,\psi_p(x)$ satisfy $G(x)\leqq\psi_i(x)\leqq F(x)$, there are sequences of partial sums tending to each $\psi_i(x)$ almost everywhere; and if any sequence of partial sums converges on a set of positive measure, the limit is almost everywhere one of the $\psi_i(x)$.

No proofs are given, but the author states a lemma on trigonometric sums on which the theorems depend. *R. P. Boas, Jr.* (Providence, R. I.).

Sheffer, I. M. Some limit theorems. Bull. Amer. Math. Soc. 54, 219–231 (1948).

The following result is a typical example of the various generalizations of the classical Cantor-Lebesgue theorem [Hardy and Rogosinski, Fourier Series, Cambridge University Press, 1944, theorem 92; these Rev. 5, 261] which the author considers. Let $\{a_{s,n}\}$, $s=1,\cdots,k$, be real or complex sequences, and let the real sequences $\{q_{s,r,n}\}$, $s=1,\cdots,k$; $r=1,\cdots,p$, be such that, for some value $r=\omega$, none of the sequences $\{q_{s,\omega,n}-q_{\sigma,\omega,n}\}$ $(s\neq\sigma)$ has zero as a limit point. If

$$\sum_{s=1}^{k} a_{s,n}\exp\{i\sum_{r=1}^{p} q_{s,r,n}x_r\}\to 0,$$

for all $(x)\equiv(x_1,\cdots,x_p)$ on a set E of positive measure, then $a_{s,n}\to 0$ $(s=1,\cdots,k)$. *K. Chandrasekharan.*

Hartman, Philip. Multiplicative sequences and Töplerian (L^2)-bases. Duke Math. J. 14, 755–767 (1947).

Consider $\phi(t)=\sum a_n\sin nt$ and write $\psi(z)=\sum a_n n^{-z}$. A general problem, (1), is that of determining when $\{\phi(nt)\}$ is a complete set in $L_2(0,\pi)$. We refer to $\psi(z)$ as complete when $\{\phi(nt)\}$ is. The results and proofs of this paper can be best appreciated by pointing out that the special case of (1) considered here is that for which (2) $\psi(z)=\prod_p\psi_p(z)$, where $\psi_p(z)=1+\sum a_{p^\nu}p^{-\nu z}$, and $\psi(z)=(1-a_p p^{-z})^{-1}$ for (3). Here p runs through prime values. The key result for (2) is that under suitable regularity conditions on $\{a_n\}$ completeness of $\psi_p(z)$ for each p is equivalent to completeness of $\psi(z)$. Hence the paper is devoted in the main to treating completeness of $\psi_p(z)$. With $w=p^{-z}$, $\psi_p(z)$ is a power series convergent for $|w|<1$, and is referred to as $\theta(w)$. Some results are stated for $(\theta(w))^{-1}$ unbounded in $|w|<1$. Results of this sort are given by the reviewer [Trans. Amer. Math. Soc. 60, 478–518 (1946); these Rev. 8, 512], for the general problem (1). [In this direction the reviewer suggests the question as to whether discreteness or perhaps zero measurability of the set of zeros of $\psi(z)$ for $\Re(z)=0$ is sufficient, granting there are no zeros for $\Re(z)>0$. The question of necessary and sufficient conditions for $\psi(z)$ to be complete in terms of properties of $\psi(z)$ as $\Re(z)\to 0$ is still open. Indeed it is not ruled out that mere existence of $(\psi(z))^{-1}$ for $\Re(z)>0$ is the answer.] *D. G. Bourgin* (Urbana, Ill.).

Pollard, Harry. The mean convergence of orthogonal series. II. Trans. Amer. Math. Soc. 63, 355–367 (1948).

[For part I cf. the same Trans. 62, 387–403 (1947); these Rev. 9, 280.] A set of sufficient conditions is deduced which are imposed on the weight-function $w(x)$ of polynomials $p_n(x)$, orthonormal on $(-1,1)$, and which ensure the mean p-convergence $(1<p<\infty)$ of the Fourier expansions of all

measurable functions $f(x)$ for which the integral

$$(1) \qquad \int_{-1}^{1} |f(x)|^p w(x)dx$$

exists. The mean p-convergence is defined by

$$(2) \qquad \lim_{N\to\infty} \int_{-1}^{1} |f(x) - \sum_{n=0}^{N} A_n^{(f)} p_n(x)|^p w(x)dx = 0,$$

where $A_n^{(f)}$ are the Fourier coefficients of the function $f(x)$. Applied to $w(x)=(1-x^2)^{s-\frac{1}{2}}$, the corresponding $p_n(x)$ then being ultraspherical polynomials, this set of sufficient conditions is shown to be verified for all values of the parameter p which belong to the interior of the interval (3) $[2-1/(s+1),\ 2+1/s]$. This result bridges the gap between two known particular cases: for Chebyshev polynomials $[s=0, w(x)=(1-x^2)^{-\frac{1}{2}}]$ (2) is valid under the condition (1) for all $p>1$ and for Legendre polynomials $[s=\frac{1}{2}, w(x)\equiv 1]$ the same is true only if $4/3<p<4$. The weight function e^{-x^2} is the limit of $(1-x^2/s)^{s-\frac{1}{2}}$ for $s\to\infty$ and the corresponding $p_n(x)$ are those of Hermite, orthonormal on $(-\infty, \infty)$. Therefore (3) suggests that in this case the only value of p for which the condition (1) entails the theorem (2) is $p=2$. This fact, as well as the similar fact for Laguerre polynomials, are confirmed by counterexamples. *E. G. Kogbetliantz* (New York, N. Y.).

Bernštein, S. N. Limit laws of the theory of best approximation. Doklady Akad. Nauk SSSR (N.S.) **58**, 525–528 (1947). (Russian)

The author continues his investigations on connections between $E_n[f(x); \lambda]$, the best approximation to $f(x)$ by polynomials of degree n on $(-\lambda, \lambda)$, and $A_p[f(x)]$, the best approximation on $(-\infty, \infty)$ by entire functions of exponential type p. In an earlier note [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 475–478 (1946), theorem 7; these Rev. **8**, 509] he proved that under certain circumstances $\lim_{n\to\infty} E_n[f(x); n/p] = A_p[f(x)]$. Now he proves that if, for some finite p_0, $A_p[f(x)]<\infty$ for $p>p_0$, then $\lim_{n\to\infty} E_n[f(x); n/(p\pm 0)] = A_{p\pm 0}[f(x)]$ for $p>\gamma p_0$, where $\gamma^{-1}(1+\gamma^2)^{\frac{1}{2}} = \log(\gamma+(\gamma^2+1)^{\frac{1}{2}})$; γ is about 1.51; in general, this value of γ is the smallest possible.

He also announces an analogue of theorem 7 for integral approximation. Let $C_n^{(q)}[f(x); \lambda]$ be the minimum of $\int_{-\lambda}^{\lambda} |f(x)-P_n(x)|^q dx$ for polynomials $P_n(x)$ of degree n, $A_p^{(q)}[f(x)]$ the minimum of $\int_{-\infty}^{\infty} |f(x)-g_p(x)|^q dx$ for entire functions $g_p(x)$ of exponential type p. Then if there is an entire function $H_0(x)$ of genus 0 such that $\int_{-\infty}^{\infty} |f(t)|^q dt < |H_0(x)|$, $\lim_{n\to\infty} C_n^{(q)}[f(x); n/(p\pm 0)] = A_{p\pm 0}^{(q)}[f(x)]$.

R. P. Boas, Jr. (Providence, R. I.).

Bernštein, S. N. A second note on homogeneous functional classes. Doklady Akad. Nauk SSSR (N.S.) **59**, 1379–1384 (1948). (Russian)

In an earlier note [same Doklady (N.S.) **57**, 111–114 (1947); these Rev. **9**, 235] the author discussed relationships between the classes Ω_q defined in terms of approximation by entire functions of exponential type and the generalized Lipschitz classes $S_q(p_i, \alpha_i, M)$ [see the cited review for definitions]. Here the author gives some generalizations and consequences of his previous results and some extensions to functions of two variables. Among others the following results are proved. (1) If $f(x)$ belongs to a class $S_q(p_i, \alpha_i, M)$, it belongs to a class Ω_q and $\sup n^q A_n f(x)$ for all $f(x)$ in S_q is equal to $Mc(S_q)$, where $c(S_q)$ depends only on the class S_q and not on n. (2) If $S_q(p_i, \alpha_i)$ has characteristic $m>q$, it contains every class $S_q(p_i', \alpha_i')$ of characteristic $m'\leqq m$. (3) Any two S_q's with the same characteristic are equivalent. (4) If $f(x)$ belongs to a class S_q of characteristic $k+1\geqq q$, then $f^{(i)}(x)\in S_{q-i}$ with characteristic $k+1$ for every integer $i<q$. (5) If $f(x)$ satisfies $A_0 f(x)<\infty$ and $\limsup_{x\to\pm\infty} |f(x)|x^{-m}<\infty$, then $f\in\Omega_q$ is equivalent to $f\in S_q$ provided that the characteristic m exceeds q.

For functions $f(x, y)$, the author defines the class $xS_{\lambda_1}^{(m_1)}(p_i, \alpha_i, M_1)\cdot yS_{\lambda_2}^{(m_2)}(q_j, \beta_j, M_2)$ by the condition that $f(x, y)$ belongs to $S_{\lambda_1}^{(m_1)}(p_i, \alpha_i, M_1)$ with respect to x and to $S_{\lambda_2}^{(m_2)}(q_j, \beta_j, M_2)$ with respect to y; the upper index is the characteristic. For a function of this class he shows that

$$A_{n_1 n_2} f(x, y) < M_1 c_{\lambda_1} n_1^{-\lambda_1} + M_2 P c_{\lambda_2} n_2^{-\lambda_2},$$
$$A_{n_1 n_2} f(x, y) < M_1 Q c_{\lambda_1} n_1^{-\lambda_1} + M_2 c_{\lambda_2} n_2^{-\lambda_2},$$

where $A_{n_1 n_2} f$ is the best approximation to $f(x, y)$ by entire functions of exponential type n_1 in x and n_2 in y, $P=\sum|p_i|$, $Q=\sum|q_i|$ and c_{λ_1}, c_{λ_2} are certain constants. As an extension of (4) [above] the author gives the theorem that, if $f(x, y)\in xS_{\gamma+\delta}^{(\gamma+m)}\cdot yS_{\gamma+\delta}^{(\gamma+m)}$ with $m>\delta>0$, then $\partial^r f/\partial x^k \partial y^l \in xS_\delta^{(\gamma+m)}\cdot yS_\delta^{(\gamma+m)}$. If $f(x, y)$ is bounded, $\partial^r f/\partial x^k \partial y^l \in xS_\delta^{(m')}\cdot yS_\delta^{(m')}$ for all $m'>\delta$. *R. P. Boas, Jr.*

Nikol'skiĭ, S. A generalization of a theorem of S. N. Bernšteĭn to differentiable functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **59**, 1533–1536 (1948). (Russian)

Let $H^{(p)}$ denote the class of functions $f(x)$ defined as follows: if $p>0$ is not an integer and $r=[p]$, then $f^{(r)}(x)$ satisfies a Lipschitz condition of order $p-r$; if p is a positive integer and $r=p-1$, then $|f^{(r)}(x+h)-2f^{(r)}(x)+f^{(r)}(x-h)|\leqq M|h|$, i.e., $f^{(r)}(x)$ is "smooth." The author proves the following theorem, the first part of which generalizes the last theorem of the paper of Bernstein reviewed above. Let $f(x, y)$ be everywhere defined and bounded, and belong to $H^{(p)}$ in x, uniformly with respect to y, and to $H^{(q)}$ in y, uniformly with respect to x. Let $p\leqq q$, $r=[p]$, and let l be an integer such that $0<l<r$. Then (1) $\partial^l f/\partial x^l$ belongs, with respect to y, to $H^{(k_*)}$ with $k_*=q(1-l/p)$, or equivalently $\partial^{l+k} f/\partial x^l \partial y^k$ belongs with respect to y to $H^{(\gamma)}$, uniformly with respect to x, where $\gamma=k_*-k$, and $k=[k_*]$ if k_* is not an integer, $k=k_*-1$ if k_* is an integer; (2) there exist functions $f(x, y)$, satisfying the hypotheses, but such that, for some x, the derivative $\partial^l f/\partial x^l$ does not belong to $H^{(k_1)}$ with respect to y for any $k_1>k_*$. The counterexample for (2) is furnished by $f(x, y)=\sum 3^{-n\delta}\cos 3^{(q/p)-\delta}x\cos 3^n y$ for l and k even.

R. P. Boas, Jr. (Providence, R. I.).

Richards, Paul I. General impedance-function theory. Quart. Appl. Math. **6**, 21–29 (1948).

The impedance function for linear electromagnetic theory is taken as defined by $\mathfrak{L}E(t)/\mathfrak{L}C(t)$, where $\mathfrak{L}$ is a Laplace transform, $E(t)$ the effect (response) at time t, $C(t)$ the cause (applied quantity) at time t. This is an instance of the Campbell and Foster definition [Fourier Integrals for Practical Applications, New York, 1931, 1948, p. 24; these Rev. **9**, 346] of the impedance of linear physical systems, namely $\mathfrak{M}E(g)/\mathfrak{M}C(g)$, which differs in the replacement of $\mathfrak{L}$ by $\mathfrak{M}$ the mate or Fourier transform, and of t by g, the epoch which is replaceable both by t and by a space variable. The definition is shown to lead to properties much like those of circuit impedance functions; in particular, many properties of driving-point impedances have their natural generalizations as indicated by a summary in physical terms of a previous paper by the author [Duke Math. J. **14**, 777–786 (1947); these Rev. **9**, 181]. *J. Riordan.*

González Domínguez, Alberto. Notes on the mathematical theory of linear circuits. I. A method for the synthesis of impedances. Math. Notae **7**, 156–161 (1947). (Spanish)

An impedance function $I(p)$, $p=t+i\omega$, may be represented to any desired degree of accuracy by the Cesàro mean $(C, 1)$ of the series $\frac{1}{2}c_0+\sum c_n[(1-p)(1+p)^{-1}]^n$ with

$$c_n=2\pi^{-1}\int_0^\pi \cos n\theta R(-\tan\theta/2)d\theta,$$

$R(i\omega)$ being the real part of the impedance function along the imaginary axis; $R(\omega)$ is required to be defined and continuous and to satisfy a Lipschitz condition over the interval $(-\infty, \infty)$. A similar formula appears when the given information is the imaginary part $S(i\omega)$ of the impedance function. As each Cesàro mean is a rational function of p, corresponding physical structures may be determined by known methods, e.g., those of Brune [J. Math. Physics **10**, 190–236 (1931)]. *J. Riordan* (New York, N. Y.).

Rozet, T. A. On inversion formulas for a class of integral transforms. Doklady Akad. Nauk SSSR (N.S.) **57**, 227–230 (1947). (Russian)

The author generalizes the well-known formula of Watson [cf. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 94]. Suppose $\omega(z)$ is a complex-valued function such that $p_1\leqq\Re\omega(c+it)\leqq p_2$ for some c and all t. Suppose further that $K(c+it)\in L^2(-\infty\leqq t\leqq\infty)$ and that $k(x, y)$ is the function whose Mellin transform is $y^{\omega(s)}K(s)$. Then for all functions $f(x)$ such that, for some $\epsilon>0$, $(x^{p_1+\frac{1}{2}-\epsilon}+x^{p_2+\frac{1}{2}+\epsilon})f(x)\in L^2$, $0\leqq x\leqq\infty$, the function

$$g(x)=\int_0^\infty k(x, y)f(y)dy$$

exists, $x^{c-\frac{1}{2}}g(x)\in L^2$ and, if $F(s)$ and $G(s)$ are the Mellin transforms of $f(x)$ and $g(x)$, respectively, then

$$G(s)=K(s)F(1+\omega(s)).$$

The author's second theorem gives a formula for $f(x)$ in terms of $g(x)$. The paper concludes with an example involving Bessel functions. *A. C. Offord* (London).

Schwartz, Laurent. Théorie générale des fonctions moyenne-périodiques. Ann. of Math. (2) **48**, 857–929 (1947).

Let E be a topological vector space over the field of complex numbers, and T a set of linear transformations t of E into itself. An element $e\in E$ is called mean-periodic if it generates an invariant manifold Te (the closure of the set of finite linear combinations of elements te) which is a proper subset of E. A set $A\subset E$ is mean-periodic if the invariant manifold $TA\neq E$. This general and intrinsic definition is shown to include Delsarte's original definition of a mean-periodic function when we specialize E to be the space of complex continuous functions $f(x)$ on the real axis $-\infty<x<\infty$ with topology given by uniform convergence on every finite interval and T the group of translations. The fundamental theorem for this space states that every (closed) invariant manifold V is characterized by its spectrum Λ, which is the set of pairs (λ, p) such that $x^{p-1}\exp(2\pi i\lambda x)\in V$ while $x^p\exp(2\pi i\lambda x)\bar{\in}V$; the λ here ranges over all complex numbers and the multiplicity p over the set of positive integers (but must be $+\infty$ for all complex λ if $V=E$).

The author proves this theorem by showing that all mean-periodic functions $f(x)\in E$ (which include all elements of V if $V\neq E$) are limits of linear combinations of exponential monomials of Tf. In fact, the spectrum of Tf is a discrete countable completely linearly independent set (as it is for any invariant manifold $V\neq E$); and f has a formal series

$$f(x)\sim\sum_{\nu,k} d_{\nu,k}(2\pi ix)^k\exp(2\pi i\lambda_\nu x),$$

$$(\lambda_\nu, p_\nu)\in\Delta_f,\ 0\leqq k<p_\nu.$$

This series is characteristic of $f(x)$ and is Abel summable to it when the terms are suitably grouped. Analytic mean-periodic functions are also defined and similar results obtained.

The author also deals with other spaces E, such as the spaces $L_1(G)$, $L_2(G)$, $L_\infty(G)$, where G is a locally compact topological Abelian group with additive notation and T is the group of translations over G. Here the results of Wiener, Beurling and Godement on closures of translations fit significantly into this theory and terminology. In particular, it is shown that a necessary and sufficient condition that a function $f\in L_1(G)$ be mean-periodic in the $L_1(G)$ sense is that its Fourier transform vanish at least once. The principal results of this paper were previously announced by the author [C. R. Acad. Sci. Paris **223**, 68–70 (1946); these Rev. **8**, 158]. *R. H. Cameron* (Minneapolis, Minn.).

Polynomials, Polynomial Approximations

Walsh, J. L. On the critical points of functions possessing central symmetry on the sphere. Amer. J. Math. **70**, 11–21 (1948).

The author determines the critical points (zeros of the derivative) of a rational function $f(z)$ whose zeros and poles occur in pairs of diametrically opposite points on a sphere Σ. If P is any point of Σ not a multiple zero of $f(z)$ and if S is the hemisphere having P as pole and containing P, then P is not a critical point if either of the following is true. (1) There is a great circle through P which separates all the zeros in S not on C from all the poles in S not on C. (2) There is a small circle C_1 having P as pole and containing P such that all the zeros in S lie exterior to C_1 and all the poles in S lie in a circular region C_2 comprised of interior points of C_1 but not containing P. These results are proved by identifying the critical points of $f(z)$ with the equilibrium points in the field of force (inverse distance law) due to repulsive particles at the zeros of $f(z)$ and attractive particles at its poles. These results are used to determine the spherical zones within which the critical points may lie when the zeros and poles lie in the pairs of diametrically opposite circular regions C_1, C_3 and C_2, C_4, respectively. They are also applied to locate the critical points of a function U which is harmonic in a spherical region possessing central symmetry and which has the value zero on one symmetric part of the boundary B of R and the value one on the remainder of B. The article closes with an interpretation of the results from the point of view of hyperbolic geometry. *M. Marden* (Milwaukee, Wis.).

Neĭmark, Yu. I. The structure of the D-decomposition of a space of polynomials and the diagrams of Vyšnegradskiĭ and Nyquist. Doklady Akad. Nauk SSSR (N.S.) **59**, 853–856 (1948). (Russian)

The paper deals with the problem of decomposing the space of the coefficients of nth degree complex polynomials

into sets corresponding to polynomials which have k roots to the right of the imaginary axis and $n-k$ roots to the left. The solution of the problem, according to the author, makes it possible to give a new treatment to the criterion of Nyquist [Bell System Tech. J. 11, 126–147 (1932)] and gives a simpler method for the construction of the diagram of Vyšnegradskiĭ. Let R_{2n} be the projective space of the polynomials

$$P_n=(a_0+ib_0)z^n+(a_1+ib_1)z^{n-1}+\cdots+(a_n+ib_n)$$

and let $D(k, n-k)$ be the set of polynomials having k roots to the left of the imaginary axis and $n-k$ to the right. This is called the D-decomposition of R_{2n}. The following results are stated. (a) Each of the sets $D(k, n-k)$ is homeomorphic to a $(2n)$-dimensional Euclidean space; (b) the boundary L_{2n-1} of all the regions $D(n, 0), \cdots, D(0, n)$ consists of polynomials which have a pure imaginary root. The remaining surfaces $\Delta_0=0, \cdots, \Delta_{2n}=0$ nowhere form boundaries of the region D. Their role consists in separating regions for which $\Delta_{2n}>0$ from others which satisfy the same conditions. The boundary of the D-decomposition of the space of real polynomials breaks up into the hyperplanes $a_0=0$ and $a_n=0$, which correspond to the infinite and zero roots, and the part of the surface $\Delta_{n-1}=0$ which corresponds to the two imaginary roots.

The plane sections formed by cutting the D-decomposition R_{2n} with the (τ, ν)-planes of the polynomials $\tau P(z)+\nu Q(z)+R(z)$ have a boundary Γ whose points satisfy the equations

$$\tau P_1(\omega)+\nu Q_1(\omega)+R_1(\omega)=0, \quad \tau P_2(\omega)+\nu Q_2(\omega)+R_2(\omega)=0,$$

where ω is real. From this it follows that Γ consists of (a) an algebraic curve whose parametric equations are

$$\tau=(Q_1R_2-Q_2R_1)(P_1Q_2-P_2Q_1)^{-1}=\Delta_\tau\Delta^{-1},$$
$$\nu=(-P_1R_2+P_2R_1)\Delta^{-1}=\Delta_\nu\Delta^{-1};$$

(b) straight lines $M\omega_j$ whose equations are

$$\tau P(i\omega_j)+\nu Q(i\omega_j)+R(i\omega_j)=0,$$

where ω_j are the real roots common to the equations $\Delta_\tau\Delta_\nu=\Delta=0$. Each of the regions into which Γ breaks up the (τ, ν)-plane consists of polynomials having the same number of roots to the left and right of the imaginary axis.

When applied to Nyquist's diagram the method not only establishes stability or instability of the given system, but also indicates the region of stability. A similar application is made to Vyšnegradskiĭ's diagram. *S. D. Zeldin.*

Fichera, Gaetano. Alcune osservazioni sulle condizioni di stabilità per le equazioni algebriche a coefficienti reali. Boll. Un. Mat. Ital. (3) **2**, 103–109 (1947).

Routh's theorem concerning algebraic equations whose roots have negative real parts is derived from a theorem by Obreschkoff [Math. Z. **45**, 747–750 (1939); these Rev. **1**, 193]. A method is described for dealing with practical cases, in particular, the characteristic equation of a differential equation whose coefficients depend on a parameter.

O. Todd-Taussky (London).

Anghelutza, Th. Sur la détermination de l'indice d'une fonction rationnelle. Acad. Roum. Bull. Sect. Sci. **28**, 265–269 (1946).

By use of Sturm sequences, the author derives the conditions of Hermite and Hurwitz for a polynomial to have all its zeros in a half-plane. *M. Marden.*

Bernšteĭn, S. N. On certain elementary extremal properties of polynomials in several variables. Doklady Akad. Nauk SSSR (N.S.) **59**, 833–836 (1948). (Russian)

Some inequalities of Markov which hold for polynomials in one variable are extended to polynomials in several variables. Among them, for instance, is the following. If x_h, $1\leqq h\leqq k$, are k independent variables and P denotes a polynomial of degrees n_h in x_h, respectively, so that, for $0\leqq s_h\leqq n_h$ and $1\leqq h\leqq k$,

$$P\equiv\sum_{1\leqq s_h\leqq n_h}A_{s_1,\cdots,s_k}x_1^{s_1}\cdots x_k^{s_k}/s_1!s_2!\cdots s_k!,$$

then the moduli of the coefficients A are bounded by the inequalities

$$|A_{s_1,\cdots,s_k}|\leqq M\prod_{h=1}^{k}|B_{s_h}^{(n_h)}|.$$

Here M is the upper bound of $|P|$ in the domain $|x_h|\leqq 1$, $1\leqq h\leqq k$, and $B_h^{(n_h)}$ is the coefficient of $x^{s_h}/s_h!$ in the expansion of the polynomial

$$T_m(x)\equiv\cos[m\cos^{-1}x]\equiv\sum_{i=0}^{m}B_i^{(m)}x^i/i!,$$

where $m=n_h$ if n_h-s_h is even and $m=n-1$ if n_h-s_h is odd.

E. G. Kogbetliantz (New York, N. Y.).

Szegö, G. On an inequality of P. Turán concerning Legendre polynomials. Bull. Amer. Math. Soc. **54**, 401–405 (1948).

For the Legendre polynomials $\{P_n(x)\}$ Turán's result is this:

(1) $$\Delta_n(x)\equiv[P_n(x)]^2-P_{n-1}(x)P_{n+1}(x)\geqq 0, \qquad n\geqq 1;\ -1\leqq x\leqq 1,$$

with equality holding only at $x=\pm 1$. Four proofs of (1) are given. Proof I uses the second order recurrence formula for $\{P_n(x)\}$ and Mehler's integral for $P_n(\cos\theta)$. In II, $\Delta_n(x)$ is expanded in a (finite) Legendre series, and by using an expression for the integral of the product of Legendre polynomials it is shown that the minimum of $\Delta_n(x)$ is $\Delta_n(1)=0$. In III the generating function $\sum_0^\infty P_n(x)z^n/n!=e^{xz}J_0(u)$, with $u=(1-x^2)^{\frac{1}{2}}z$, is shown to belong to the class of entire functions considered by Pólya and Schur [J. Reine Angew. Math. **144**, 89–113 (1914)], from which fact (1) follows (also, in the same way, the analogue of (1) holds for ultraspherical, Laguerre and Hermite polynomials). Finally, in IV, the transcendental elements in III are dispensed with, and (1) is shown to be a consequence of the fact that the polynomial $\sum_{k=0}^{n}\binom{n}{k}P_k(x)z^k$ has, in z, only real roots.

I. M. Sheffer (State College, Pa.).

Geronimus, Ya. L. On asymptotic formulas for orthogonal polynomials. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 3–14 (1948). (Russian)

This paper is essentially a study of the relation between the weight-function of polynomials $P_n(z)$, orthogonal on the circumference $z=e^{i\theta}$ with respect to the distribution $d\sigma(\theta)$ (where $\sigma(\theta)$ is a bounded, nondecreasing function which is not constant over an infinite point set in $(0, 2\pi)$), and the sequence (a_n) of Schur's parameters of the family $[P_n(z)]$ in the recurrent relations

(R) $$P_{n+1}=zP_n(z)-\bar{a}_nz^nP_n(z^{-1}), \qquad |a_n|<1.$$

The validity of known asymptotic formulas for these polynomials [see G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 290, theorems 12.1.1, 12.1.3; these Rev. **1**, 14], which

were established under the hypothesis that $\sigma(\theta)$ is an absolutely continuous function, is extended beyond this condition. The extension is based on the recurrent relations (R), the behavior of the sequence (a_n) being related to properties of the function $\sigma(\theta)$.

The main result states that a necessary and sufficient condition for the validity in question is

$$(C)\quad \lim_{n\to\infty}\{\overline{p(z)}\sum_{k=n}^{\infty}a_kz^k+\sum_{k=0}^{\infty}\sum_{s=n}^{\infty}A_{nks}[p(z)+\overline{p(z)}\sum_{i=s}^{\infty}a_iz^i]\}=0,$$

where $|z|=1$, $p(z)=(2\pi)^{\frac{1}{2}}D^{-1}(z)$ in Szegö's notation and A_{nks} are defined by recurrence: $A_{n0s}=\bar{a}_sz^{s+1}\sum_{m=n}^{s}a_mz^{m+1}$ and $A_{n,k+1,s}=\sum_{m=n}^{s}A_{nkm}A_{m0s}$, $n<s$. If only a sufficient condition is to be formulated, (C) takes the form (C′): the series $\sum_0^\infty a_nz^n$ converges for $|z|=1$ and $\lim_{n\to\infty}\sum_{i=n}^{\infty}\sum_{i=n}^{\infty}|A_{nks}|=0$, where $|z|=1$, or even a simpler form (C″): the series $\sum_0^\infty a_nz^n$ converges for $|z|=1$ and, also for $|z|=1$,

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n+k}|\,|a_n+a_{n+1}z+\cdots+a_{n+s}z^s|=0.$$

E. G. Kogbetliantz (New York, N. Y.).

Jackson, Dunham. The boundedness of orthonormal polynomials on certain curves of the fourth degree. Trans. Amer. Math. Soc. **63**, 193–206 (1948).

The monomials $x^\alpha y^\beta$ are arranged according to increasing $\alpha+\beta$ and for a fixed $\alpha+\beta$ according to increasing β. Assuming a relation $y=Ax^4+Bx^3+Cx^2+Dx+E$, $A\neq0$, certain monomials can be eliminated and applying the Schmidt process orthogonal polynomials on this algebraic curve are formed. These polynomials are expressible in terms of Legendre polynomials and their boundedness on a fixed arc is shown. A similar conclusion is reached in various other cases of curves of the fourth degree. In the case of the curve $x^4+y^4=1$ boundedness holds uniformly on every arc except near the points where the curve intersects the coordinate axes. *G. Szegö* (Stanford University, Calif.).

Gröbner, Wolfgang. Über die Konstruktion von Systemen orthogonaler Polynome in ein- und zwei-dimensionalen Bereichen. Monatsh. Math. **52**, 38–54 (1948).

It is shown how orthogonal polynomial systems can be obtained as the solution of a minimum problem with auxiliary conditions, which is resolved by a method borrowed from the calculus of variations. The following generalization of a result of Szegö [Math. Ann. **82**, 188–212 (1920), in particular, pp. 196–198; Math. Z. **9**, 218–270 (1921), in particular, pp. 234–235] for certain polynomial sequences is proved and utilized. In order that the set of functions $\{\varphi_n(x)\}$ be orthogonal with respect to the non-negative weight function $Q(x)\not\equiv0$ on $a\leqq x\leqq b$, it is necessary and sufficient that for each n $(=1, 2, \cdots)$ the integral $J_n=\int_a^b Q\{c_1\varphi_1+\cdots+c_{n-1}\varphi_{n-1}+\varphi_n\}^2dx$ be minimized by the choice $c_1=\cdots=c_{n-1}=0$. The lemma has an extension to more than one variable.

As example of the method, let $\{P_n(x)\}$, with P_n of degree n and subject to condition (1) $P_n^{(n)}(x)=n!$, be orthogonal on $[a, b]$ with weight function $Q(x)\equiv1$. By the above lemma, P_n minimizes (2) $J_n=\int_a^b\{P_n(x)\}^2dx$. By the Euler-Lagrange method, the expression

$$\Omega_n=\int_a^b\{P_n^2+\lambda_n(P_n^{(n)}-n!)\}dx$$

is set up, where λ_n is an arbitrary function, and its variation $\delta\Omega_n$ made to vanish. From this follow the relations $P_n=\frac{1}{2}(-1)^{n-1}\lambda_n^{(n)}$, $\lambda_n^{(2n)}=2(-1)^{n-1}n!$ and $\lambda_n=\lambda_n^{(1)}=\cdots=\lambda_n^{(n-1)}=0$ for $x=a$ and $x=b$. This leads to the solution

$$P_n=\frac{n!}{(2n)!}\frac{d^n}{dx^n}[(x-a)^n(x-b)^n],\quad n=0, 1, \cdots,$$

which, except for a constant multiplier, is the Legendre polynomial for $[a, b]$.

By altering condition (1) and by using other weight functions (thus changing (2)), the classical Jacobi, Laguerre and Hermite polynomials can be obtained, as well as others. The method extends to two and more variables, at least for simple regions (to replace the interval $[a, b]$), and the work is carried out for the cases of a triangle and an ellipse. The details cannot be gone into in a brief review.

I. M. Sheffer (State College, Pa.).

Mursi, M., et Makar, R. H. Sur la base inverse d'une base de polynomes. Bull. Sci. Math. (2) **71**, 47–51 (1947).

Using J. M. Whittaker's terminology [Interpolatory Function Theory, Cambridge University Press, 1935], we call a set of polynomials $p_n(z)$ basic if the (row-finite) matrix of its coefficients has a row-finite reciprocal. The elements of the reciprocal matrix can then be taken as the coefficients of a new basic set $q_n(z)$ [the authors' "base inverse"]. The authors prove the following theorems. If $\{p_n\}$ is a simple set (degree of p_n is n) and if the coefficient of z^n in $p_n(z)$ is 1, then $\{q_n\}$ is effective on $|z|=R$ if $\{p_n\}$ is. If $\{p_n\}$ is of order ω, the order $\tilde{\omega}$ of $\{q_n\}$ satisfies $\frac{1}{2}\omega\leqq\tilde{\omega}\leqq2\omega$, with equality possible. *R. P. Boas, Jr.*

Gavurin, M. K. On the approximation of a continuous function by a linear differential transform of a polynomial. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 15–30 (1948). (Russian)

Write $D=\sum_{j=0}^{k}\varphi_j(x)(d/dx)^{k-j}$ (φ_j continuous, $|\varphi_0(x)|+|\varphi_1(x)|>0$ in $<a, b>$). Let E be the set of x in $<a, b>$ where $\varphi_0(x)=0$, $E_1=CE\cap<a, b>$. S. N. Bernšteĭn studied the problem under which conditions the "D-polynomials" $DP(x)$ ($P(x)$ a polynomial) are dense in $C<a, b>$ [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 15–42 (1941); these Rev. **2**, 363]. He defined a "regular solution" of (1) $Dy=A(x)$ ($A(x)$ continuous) by the conditions (a) $y, y', \cdots, y^{(k-1)}\in C<a, b>$, (b) $y^{(k)}$ is continuous in E_1, (c) (1) is satisfied in $<a, b>$, where $\varphi_0(x)y^{(k)}(x)$ is interpreted as 0 for $x\in E$. Bernšteĭn's results were (α) a necessary and sufficient condition for the D-polynomials to be dense in $C<a, b>$ is that (1) has a regular solution for every $A(x)\in C<a, b>$; (β) if (1) has a regular solution for $A(x)=A_0(x)\in C<a, b>$, then $A_0(x)$ can be uniformly approximated by D-polynomials; (γ) if a continuous function $A(x)$ can be uniformly approximated by D-polynomials, then (1) has a regular solution.

In the present paper the author points out that (β) is false and has to be replaced by the theorem: $A_0(x)\in C<a, b>$ can be uniformly approximated by D-polynomials, if and only if (i) (1) has at least one regular solution for $A(x)=A_0(x)$; (ii) in every subinterval $<\alpha, \beta>$ of $<a, b>$ in which $\int_\alpha^\beta|\varphi_0(x)|^{-1}dx<\infty$, $y^{(k-1)}$ is absolutely continuous. After proving the necessity part of the theorem the author gives the following counter-example to (β): let $<a, b>=<0, 1>$, E Cantor's set, φ_0 a continuous function vanishing in E and such that $\int_0^1|\varphi_0(x)|^{-1}dx<\infty$, $D=\varphi_0d/dx+1$, $\theta(x)$ the well-

known singular function for which $\theta(0)=0$, $\theta(1)=1$, $\theta'(x)=0$ in E_1. It is easy to see that $y=\theta(x)$ is the only continuous solution of $Dy=\theta(x)$. This solution is regular. But $\theta(x)$ cannot be uniformly approximated by D-polynomials, since otherwise $\theta(x)$ would be absolutely continuous by the necessity part of the theorem. It is also shown that under the hypothesis of (α) the condition (ii) of the theorem must be satisfied, so that (α) remains true.

The proofs are based on a straightforward investigation of the real variable properties of solutions of (1). A typical lemma is: if $\varphi_1, \cdots, \varphi_k$, $A(x)\varepsilon C<a, b>$ and are of bounded variation and if y is a regular solution of (1), then $y^{(k-1)}$ is of bounded variation. *W. H. J. Fuchs* (Ithaca, N. Y.).

Special Functions

***Truesdell, C. An Essay Toward a Unified Theory of Special Functions Based upon the Functional Equation** $\frac{\partial}{\partial z}F(z,\alpha)=F(z,\alpha+1)$. Annals of Mathematics Studies, no. 18. Princeton University Press, Princeton, N. J., 1948. iv+182 pp. $3.00.

The aim of this monograph is to provide a general theory which will motivate, discover and coordinate the seemingly disconnected, and yet strikingly similar, relations among well-known families of special functions. It is the author's contention that the striking similarity is due to the fact that each family of special functions involved in the comparative study satisfies a functional equation which in all but five cases can be reduced to the equation given in the title.

Chapter I gives the object and plan of the monograph. Chapter II deals with the reduction of linear homogeneous difference-differential equations of the first order to the F-equation; gives a list of 48 solutions of the F-equation in terms of elementary functions, hypergeometric and confluent (also generalised) hypergeometric functions, incomplete gamma functions, generalized zeta functions, Bessel, Legendre, Gegenbauer, Jacobi, Hermite, Laguerre and similar functions; and shows that if, and only if, both $f(y,\alpha)$ and $f(y,-\alpha)$ satisfy functional equations which can be reduced to the F-equation, f satisfies a linear homogeneous differential (difference) equation of the second order in y (α).

In chapter III existence and uniqueness theorems are derived not only for the F-equation but also for a more general functional equation (a system of not necessarily linear difference-differential equations of any order). These theorems are applied and further developed in chapter IV in order to determine that solution of the F-equation which reduces to a given $\phi(\alpha)$ when $z=z_0$. This chapter occupies nearly half of the book and contains very numerous applications to special functions. Series in powers of $z-z_0$ are obtained and used to derive a number of special expansions (e.g., Euler's transformation of the hypergeometric series, and many generating functions). Inverse factorial and Newton series in α, and contour integral solutions of the F-equation, are further sources of special relations. Certain definite integrals are also evaluated, and a method is invented for discovering relations between various functions which are solutions of the F-equation.

The determination, in chapter V, of solutions which reduce to a given $\psi(z)$ when $\alpha=\alpha_0$ is less satisfactory on account of a lack of uniqueness inherent in the problem. In many cases the results are valid only when $\alpha=\alpha_0+m$, where m is a nonnegative integer; more general results can often be conjectured, but must be proved separately. Among the results obtained in this chapter are nth derivative formulae and some more integral formulae.

Chapter VI brings a historical note which explains that similar methods have been used by earlier writers (for instance, by Sonine and Nielsen in their study of cylinder functions); but the present essay is the first attempt at a systematic and thorough exploitation of the functional equation. The general conclusions end on a healthy note of caution. While the author maintains that his method is very powerful, he warns against its exaggerated use to the exclusion of all other approaches to the subject in hand. "A finished treatment of special functions," he says, "should rely exclusively on no one approach; were a new treatise to be written, I believe it should employ those tools which lend themselves to the most efficient and elegant deductions, and that one of those tools would be the theory of the F-equation."

Three appendices give the definitions and some properties (considered as given) of the special functions that occur in the book, the definitions of various operators, and the five cases of the functional equation $\partial g(y,\alpha)/\partial y=C(y,\alpha)g(y,\alpha+1)$ which cannot be reduced to the F-equation. There is also a bibliography.

A brief summary of the principal results has appeared earlier [Proc. Nat. Acad. Sci. U. S. A. **33**, 82–93 (1947); these Rev. **8**, 517]. *A. Erdélyi* (Edinburgh).

Kline, Morris. Some Bessel equations and their application to guide and cavity theory. J. Math. Physics **27**, 37–48 (1948).

A function $x(t)$, defined for positive t, which satisfies the equation (*) $J_n(x)Y_n(tx)-J_n(tx)Y_n(x)=0$ identically in t, is called a solution of (*); n is a nonnegative integer. The author proves that to each positive zero s_{nm} of $J_n(x)$ there is a unique continuous solution of (*) which approaches s_{nm} as $t\to+0$. Also, as $t\to+0$, each continuous solution of (*) approaches a zero of $J_n(x)$; and the correspondence between zeros of $J_n(x)$ and solutions of (*) is unique. Similar results hold when all Bessel functions appearing here are replaced by their derivatives. The results are applied to electromagnetic wave guide theory. *A. Erdélyi* (Edinburgh).

Cârstoiu, Ion. Applications nouvelles du calcul symbolique aux fonctions de Bessel. C. R. Acad. Sci. Paris **226**, 769–770 (1948).

The method and results are known. [Cf. E. T. Copson, Proc. London Math. Soc. (2) **33**, 145–153 (1932).] *A. Erdélyi* (Edinburgh).

Meixner, J. Neuere Ergebnisse über Sphäroid-Funktionen. Z. Angew. Math. Mech. **25/27**, 137–138 (1947).

The author announces various results of a study concerning the spheroidal wave equation,

$$(1-x^2)y''-2xy'+\{\lambda-\gamma^2x^2-m^2/(1-x^2)\}y=0.$$

Asymptotic series for the characteristic values λ, in descending powers of $\gamma>0$ [prolate spheroidal wave functions], are explicitly given, up to terms including γ^{-3}. The specialization obtained when $m=\frac{1}{2}$ applies to the differential equation of Mathieu. Attention is drawn to solutions y that, if m is an integer, are expressible as a finite linear combination of

algebraic functions of powers and exponentials; the relevant λ satisfies an algebraic equation of degree m. A general series expansion of spheroidal wave functions, in terms of Bessel functions and spherical harmonics, appears to include many interesting cases, some of which are known. No proofs are given. *C. J. Bouwkamp* (Eindhoven).

Bailey, W. N. A transformation of nearly-poised basic hypergeometric series. J. London Math. Soc. **22** (1947), 237–240 (1948).

The set of basic analogues of transformations of ordinary hypergeometric series is extended by the addition of a transformation of a nearly-poised series ${}_7\Phi_6$ into a ${}_{12}\Phi_1$. This is established by the usual methods of operating with closed sums as outlined in W. N. Bailey's Generalized Hypergeometric Series [Cambridge University Press, 1935]. *N. A. Hall* (Minneapolis, Minn.).

Kourganoff, Vladimir. Sur les fonctions $K_n(x)$ et certaines intégrales qui s'y rattachent. II. Calcul des intégrales $I_{p,s,n,m}(a,b)=\int_0^\infty e^{-px}x^s K_n(ax)K_m(bx)dx$. Ann. Astrophysique **10**, 329–340 (1947).

[For part I cf. same Ann. **10**, 282–299 (1947); these Rev. **9**, 349.] Detailed statement and derivation of the results announced in C. R. Acad. Sci. Paris **225**, 451–453 (1947); these Rev. **9**, 92. *A. Erdélyi* (Edinburgh).

Yakovlev, A. I. Microphone functions. Doklady Akad. Nauk SSSR (N.S.) **59**, 651–654 (1948). (Russian)

Coefficients of the Fourier series expansion of the solution of the equation $L(di/dt)+(R-r\cos\omega t)i=E$, where L, R, r, ω, E are constants, are given by means of the functions (called microphone functions)

$$\sum_{n=1}^{\infty} A_n(2n-1)!!r^{2n}/\{(2n)!!\prod_{k=1}^{n}(R^2+k^2\omega^2L^2)\},$$

where $A_n=1$, n or $n-1$. *I. Opatowski* (Chicago, Ill.).

Harmonic Functions, Potential Theory

Walsh, J. L. Note on the location of the critical points of harmonic functions. Bull. Amer. Math. Soc. **54**, 191–195 (1948).

A previous result of the author [Bull. Amer. Math. Soc. **52**, 346–347 (1946); these Rev. **7**, 382] is here extended so as to admit arcs on C_0 where the harmonic function $u(x,y)$ is also assigned the value one and so as to allow the intersection of the curves $C_1, \cdots, C_n$ with C_0. The possible choices of the region R_0 are discussed, together with the advantage of taking R_0 as large as possible. An application is made to a function $u(x,y)$ which is harmonic in a region R bounded by two disjoint Jordan curves C_0 and C_1 and by Jordan arcs or curves $C_2, \cdots, C_n$ drawn in the annulus R_0 between C_0 and C_1. If $B_1, \cdots, B_m$ denote m mutually disjoint arcs of C_1, the latter function $u(x,y)$ is further assumed to have the value zero on $C_0+B_1+\cdots+B_m$ and the value one on the remainder of the boundary, no continuity being required at end-points. It is then shown that, if $h(z, C_0, R_0)$ denotes the harmonic measure of C_0 in z relative to R_0 and if k denotes a constant not less than $\frac{1}{2}$, no critical point of $u(x,y)$ lies in the largest region $h(z, C_0, R_0)>k$ not containing any point of $C_2+\cdots+C_n$. *M. Marden.*

Walsh, J. L. The critical points of linear combinations of harmonic functions. Bull. Amer. Math. Soc. **54**, 196–205 (1948).

The linear combination first considered is $U(z)=Ag(z,t,R)+Bh(z,s_1,R)$, where A and B are real constants, $g(z,t,R)$ is the Green's function with pole at t for a Jordan region R with boundary C and $h(z,s_1,R)$ is the harmonic measure of the arc s_1 of C relative to the point z and to the region R. Regarded as the equilibrium point in a field due to the repulsion of four particles according to the inverse distance law, the critical point of $U(z)$ is found to lie on the non-Euclidean bisector of the arc s_1. This result is generalized to the linear combination $U(z)=\sum_{k=1}^{n}A_k g(z,t_k,R)+\sum_{k=1}^{m}B_k h(z,s_k,R)$, where the t_k are points of R, the s_k are arcs of C and the A_k and B_k are real constants. This combination is shown not to have any critical point in the locus of all non-Euclidean lines (if any) which separate the t_k and s_k corresponding to positive A_k and B_k from those corresponding to negative A_k and B_k. Under suitable conditions the same result holds for infinite sums and integrals of the above type. Hence it is concluded that, if $U_1(z)$ is bounded on C and continuous except for a finite number of points on C, if $U_1(z)$ is positive on arc s_1 of C, negative on arc s_2 (disjoint from s_1) of C and zero elsewhere on C, and if finally $U(z)$ is not identically constant, harmonic and bounded in R, continuous on C and equal to $U_1(z)$ at the points of continuity of $U_1(z)$ on C, then in R all the critical points of $U(z)$ lie in the sets $h(z,s_1,R)\geqq\frac{1}{2}$ and $h(z,s_2,R)\geqq\frac{1}{2}$. *M. Marden.*

Walsh, J. L. Critical points of harmonic functions as positions of equilibrium in a field of force. Proc. Nat. Acad. Sci. U. S. A. **34**, 111–119 (1948).

In various previous papers the author determined the critical points of a harmonic function $u(x,y)$ by a limiting process from known results on the zeros of the derivative of a rational function $r(z)$. In the present paper he shows how the critical points of $u(x,y)$ may be studied directly by a method which he and others have used successfully in the study of the zeros of $r'(z)$. For example, if R is an infinite region bounded by a finite Jordan curve (more generally, configuration) B and if $G(x,y)$, the Green's function for R with pole at infinity, has the harmonic conjugate $H(x,y)$, then the analytic function $F(z)=G(x,y)+iH(x,y)$ has the derivative $F'(z)=\int_B d\sigma(t)/(z-t)$. Since $d\sigma(t)=-dH/2\pi>0$ on B, the conjugate imaginary of $F'(z)$ is the force due to a mass distribution $d\sigma(t)$ on B repelling according to the inverse distance law. The critical points of $G(x,y)$ being the equilibrium points in this field are located in the smallest convex set containing B. This result is analogous to the theorems of Gauss and Lucas on the zeros of the derivative of a polynomial. A result analogous to the Jensen-Walsh theorems on the zeros of the derivative of a real polynomial is obtained for the case that B is symmetric in the axis of reals. If B consists of the mutually disjoint Jordan configurations C and D and if $u(x,y)$ is harmonic in R, continuous in $R+B$, zero on C and unity on D, then the critical points of $u(x,y)$ are the equilibrium points in a field due to a certain spread of positive mass on C and negative mass on D. With the aid of non-Euclidean geometry the same method is used to locate the critical points of a still more general class of harmonic functions. *M. Marden* (Milwaukee, Wis.).

Gergen, J. J., and Dressel, F. G. A minimal problem for harmonic functions. Duke Math. J. **14**, 889–906 (1947).

Soient $\lambda_1(\theta)$, $\lambda_2(\theta)$ deux fonctions "en escalier" données; $h_1, \cdots, h_n$ sont n nombres réels; $0\leqq\theta_1<\cdots<\theta_n=2\pi$ sont

n arguments; on pose $h_{n+1}=h_1$, $\theta_{n+1}=\theta_1$, et on suppose $\lambda_1(\theta)=\min(h_k, h_{k+1})$, $\lambda_2(\theta)=\max(h_k, h_{k+1})$ pour $\theta_k\leqq\theta<\theta_{k+1}$. L'article traite du problème suivant: étudier le minimum de l'intégral $\iint_{r<1}|\text{grad } U|^2 dS$, lorsque $U(r,\theta)$ est une fonction harmonique pour $r<1$ vérifiant $\liminf_{r\to 1} U(r,\theta)\geqq\lambda_1(\theta)$, $\limsup_{r\to 1} U(r,\theta)\leqq\lambda_2(\theta)$. Si $\max\lambda_1(\theta)\leqq\min\lambda_2(\theta)$, il y a une fonction constante ou une infinité de fonctions constantes pour lesquelles l'intégrale est nulle. Ce cas exceptionnel mis à part, il y a une fonction U minimisante et une seule, non constante, et le minimum est fini et positif. Principe de la démonstration: dans un espace de Hilbert, il y a sur un ensemble fermé convexe un point unique dont la norme est minima. Il ne s'agit pas d'une vraie norme, puisque l'intégrale est nulle pour une fonction constante, et c'est la cause du cas exceptionnel.

Une étude precise montre que, si on pose $F(z)=U(r,\theta)+iV(r,\theta)$ $(z=re^{i\theta})$, la fonction minimisante U est telle que $(zF'(z))^2=\frac{1}{4}i\sum_1^n B_k(z_k+z)/(z_k-z)$, où les B_k sont des constantes réelles, avec $\sum B_k=\sum B_k/z_k=0$. Cette formule permet de déterminer complètement $F(z)$ pour $n\leqq 4$. La démonstration utilise un calcul de variation et une formule de Douglas [Trans. Amer. Math. Soc. 33, 263–321 (1931)].

L. Schwartz (Nancy).

Deny, Jacques. Le principe des singularités positives de G. Bouligand et la représentation des fonctions harmoniques positives dans un domaine. Revue Sci. 85, 866–872 (1947).

Two important problems in modern potential theory are the following: (i) the characterization of frontier points A of a domain D which have the property that a function that is positive and harmonic in D and vanishes continuously at each frontier point, except A, of D, is determined uniquely up to a multiplicative constant; (ii) the representation of functions that are positive and harmonic in a domain D by means of integrals of Poisson-Stieltjes type. The author gives a survey of recent work on these problems.

M. Reade (Ann Arbor, Mich.).

Pfluger, Albert. Sur l'unicité de la distribution de masses produisant un potentiel donné. Bull. Sci. Math. (2) 71, 45–47 (1947).

The author gives an elementary proof of a result due to F. Riesz [Acta Math. 54, 321–360 (1930)], that the potential due to a distribution of nonnegative mass on a Borel set uniquely determines the distribution of mass.

M. Reade (Ann Arbor, Mich.).

Maruhn, Karl. Über einige Klassen nichtlinearer Randwertaufgaben der Potentialtheorie. Math. Z. 51, 36–60 (1947).

Let T be a finite domain of S_3 bounded by a closed surface S meeting certain requirements of smoothness. The first result given refers to the existence of a function u continuous in $T+S$ and harmonic in T which satisfies at each point σ of S the nonlinear boundary condition $u(\sigma)=\mathfrak{B}\{u(\sigma)\}$. Here $\mathfrak{B}$ is a function of $u(\sigma)$ continuous for σ on S and satisfying certain inequalities involving small nonnegative values of a parameter C. The method of successive approximations is used. By introducing further hypotheses uniqueness may be assured. The corresponding exterior problem is also discussed. Let $u_0(x,y,z)$ be the solution, for boundary values $f(\sigma)$, of the first boundary value problem of potential theory, for either the interior or exterior region bounded by S. The author studies the boundary value problem $\nabla^2 u=0$, $u(\sigma)=f(\sigma)+\mathfrak{B}\{u(\sigma)\}$, $\lim_{C\to 0} u=u_0$. A solution of such a problem is said to be in the neighborhood of the solution of the classical problem. Discussions are included for the interior and exterior problems for $\nabla^2 u=0$ with boundary conditions of the forms

$$(\partial u/\partial n)_\sigma=\mathfrak{B}\{u(\sigma), u_x(\sigma), u_y(\sigma), u_z(\sigma)\},$$
$$(\partial u/\partial n)_\sigma+h(\sigma)u(\sigma)=\mathfrak{B}\{u(\sigma), u_x(\sigma), u_y(\sigma), u_z(\sigma)\},$$

and solutions of $\nabla^2 u=0$ in the neighborhood of solutions of the second and third boundary value problems of the classical theory. The author also notes that an analogous theory is valid in the plane.

F. W. Perkins (Hanover, N. H.).

Kampé de Fériet, J. On a property of the Laplacian of a function in a two dimensional bounded domain, when the first derivatives of the function vanish at the boundary. Math. Mag. 21, 74–79 (1947).

Let D be a finite domain bounded by a curve B; it is assumed that B is sufficiently smooth to permit the use of Green's theorem. The author considers four function spaces: (C) the space of functions continuous in D; (R) the space of functions f for which $\partial f/\partial x$ and $\partial f/\partial y$ are continuous in $D+B$ and $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ are continuous in D; (H) the space of functions harmonic in D; (L) the subspace of (C) orthogonal to (H), that is, the space of functions f such that for every g in (H) we have $\iint_D fg\,d\sigma=0$. The principal results are contained in two theorems. (I) If $\psi\varepsilon(R)$, if ψ is constant on B and if $\partial\psi/\partial n=0$ on B, then $\nabla^2\psi\varepsilon(L)$. (II) If $\psi\varepsilon(R)$, if ψ is constant on B and if $\nabla^2\psi\varepsilon(L)$, then $\partial\psi/\partial n=0$ on B. The hydrodynamical significance of these theorems is noted.

F. W. Perkins (Hanover, N. H.).

Kampé de Fériet, Joseph. Sur la moyenne polyharmonique d'une fonction. C. R. Acad. Sci. Paris 226, 621–623 (1948).

Let D be a bounded domain in n-space, and let A_s^* denote the set of real functions which are polyharmonic of order s in D. If H is the Hilbert space of real functions with summable squares over D, then, as the author shows, the linear variety $A_s\equiv A_s^*\cap H$ is closed in H. If B_s is the closed linear variety orthogonal to A_s in H, then each $f\varepsilon H$ can be written in the form $f=f_s+g_s$, where $f_s\varepsilon A_s$ and $g_s\varepsilon B_s$. The function f_s is defined to be the polyharmonic mean, of order s, of f. A simple result is that $\min\|f-v\|$ for $v\varepsilon A_s$ is equal to $\|f-f_s\|$, and f_s is unique. The author intends to use these and similar notions in a forthcoming work on fluid mechanics.

M. Reade (Ann Arbor, Mich.).

Višik, M. I. The method of orthogonal projection for general linear self-adjoint elliptic differential equations. Doklady Akad. Nauk SSSR (N.S.) 58, 957–960 (1947). (Russian)

In preceding papers the author has applied the method of orthogonal projection in Hilbert space to boundary value problems for second order elliptic partial differential equations [C. R. (Doklady) Acad. Sci. URSS (N.S.) 56, 115–118 (1947); these Rev. 9, 93] and for the general self-adjoint elliptic equation of order $2m$ [thesis, Math. Inst. Acad. Sci. USSR, 1947]. In this paper the method is applied to boundary value problems for the biharmonic equation $\Delta\Delta u=0$ in n independent variables. The Hilbert space $H(\Delta^2)$ is the space of symmetric $n\times n$ matrices whose components are functions in L in a region G. The scalar product is

$$(*)\qquad (h,g)=\int\cdots\int_G\sum_{i_1,i_2} g_{i_1,i_2}h_{i_1,i_2}\,dx_1\cdots dx_n.$$

Let $Z(\Delta^3)$ be the closure of the set of matrices of the second partial derivatives of functions which vanish outside some subregion G' ($\bar{G}' \subset G$) and have in G continuous partial derivatives of the fourth order, $\Psi^0(\Delta^3)$ the closed linear manifold generated by matrices (ψ^0) with twice differentiable components which vanish outside a cube K ($\bar{K} \subset G$) and satisfy $\sum \partial^2\psi_{i_1 i_2}/\partial x_{i_1}\partial x_{i_2} = 0$. The orthogonal complement $U(\Delta^3)$ of $Z \oplus \Psi^0$ consists of the matrices of the second partial derivatives of solutions of $\Delta\Delta u = 0$ in cubes K. The orthogonal decomposition (**) $F(\Delta^3) = U(\Delta^3) \oplus Z(\Delta^3)$ gives the solution of a problem of Dirichlet type; (***) $\Psi(\Delta^3) = U(\Delta^3) \oplus \Psi^0(\Delta^3)$ corresponds to a generalization of the Neumann problem.

An arbitrary region G is approximated by regions G_i with "smooth" boundaries Γ_i, and the solution of the boundary value problem for G is a weak limit in a certain sense of the solutions for the regions G_i. The author states that the decomposition (**) makes it possible to show the existence of many-valued solutions of the biharmonic equation for an arbitrary region and defines the periods of such solutions. He asserts that by a different definition of the scalar product it is possible to set up an orthogonal decomposition which corresponds to a problem which is a generalization of the "mixed" problem for the Laplacian, and refers to similar results for more general elliptic equations.

M. J. Gottlieb (Princeton, N. J.).

Bremekamp, H. Construction of the solution of $\Delta^k u = 0$, satisfying given boundary conditions, on a circle or a sphere. Nieuw Arch. Wiskunde (2) **22**, 293–299 (1948).

The author considers the problem of determining the function $u(r, \theta)$ which is k-harmonic within the circle $r = 1$ and satisfies on the boundary the conditions

$$u = f_0(\theta), \quad \partial^p u/\partial r^p = f_p(\theta), \qquad p = 1, 2, \cdots, k-1,$$

where $f_p(\theta) \varepsilon C^\lambda$ for some sufficiently large λ. He sets

$$f_p(\theta) = \sum_{m=0}^{\infty} (A_{p,m} \cos m\theta + B_{p,m} \sin m\theta),$$

$$p = 0, 1, 2, \cdots, k-1;$$

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{k-1} r^{m+2k-2n-2}(a_{n,m} \cos m\theta + b_{n,m} \sin m\theta).$$

The solution of certain systems of linear algebraic equations yields unique values for the a's and b's for which u satisfies the requirements of the problem. A parallel theory is outlined for the sphere, involving spherical harmonics.

It may be noted that the existence of a solution for a similar boundary value problem for polyharmonic functions with reference to a region in S_n has been discussed by N. Meyman [C. R. (Doklady) Acad. Sci. URSS (N.S.) **33**, 275–278 (1941); these Rev. **5**, 123]. *F. W. Perkins.*

Bremekamp, H. Construction of the solution of $\Delta\Delta u = 0$, satisfying given boundary conditions, in the case that the boundary is an ellipse. Nieuw Arch. Wiskunde (2) **22**, 300–305 (1948).

The existence of a function biharmonic in a plane region and having prescribed values and prescribed normal derivative has been investigated by various authors. [See, for example, A. Korn, Ann. Sci. École Norm. Sup. (3) **25**, 529–583 (1908).] In the present paper the author introduces curvilinear coordinates $x = c \cosh \xi \sin \eta$, $y = c \sinh \xi \sin \eta$, and then seeks to determine α_n, β_n, γ_n and δ_n so that the harmonic functions

$$v = \sum_0^{\infty} e^{n\xi}(\alpha_n \cos n\eta + \beta_n \sin n\eta),$$

$$w = \sum_0^{\infty} e^{n\xi}(\gamma_n \cos n\eta + \delta_n \sin n\eta),$$

yield a function $u = (x^2+y^2)v + w$ which is biharmonic in the domain $0 \leqq \xi < \xi_0$ interior to the ellipse $\xi = \xi_0$ and which satisfies the following boundary conditions on $\xi = \xi_0$:

$$u(\xi_0, \eta) = \varphi_1(\eta) = \sum_0^{\infty} (P_{1,n} \cos n\eta + P_{2,n} \sin n\eta),$$

$$[\partial u/\partial \xi]_{\xi=\xi_0} = \varphi_2(\eta) = \sum_0^{\infty} (Q_{1,n} \cos n\eta + Q_{2,n} \sin n\eta).$$

Since $[\partial u/\partial n]_{\xi=\xi_0}$ is expressible in terms of $[\partial u/\partial \xi]_{\xi=\xi_0}$ the second boundary condition is equivalent to prescribing the normal derivative on $\xi = \xi_0$. The author finds that α_0, β_1 and β_2 are determined by the requirement that the series for v converge in the interior of the ellipse, and the other coefficients are then determined by recursion formulas obtained from the boundary conditions. *F. W. Perkins.*

Dore, Paolo. Sull'uso della seconda funzione di Green nei procedimenti di calcolo degli scostamenti del geoide dall'ellissoide a mezzo delle anomalie gravimetriche. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **8**, 177–181 (1941).

Differential Equations

Pompeiu, D. Les fonctions indéfiniment symétriques et les équations différentielles. Acad. Roum. Bull. Sect. Sci. **24**, 291–296 (1943).

The author proves some theorems regarding solutions of a few cases of an ordinary differential equation of the first order with variables separable. *A. B. Brown.*

Mangeron, D. Sur un théorème de Mr. Pompeiu relatif à une classe d'équations différentielles. Acad. Roum. Bull. Sect. Sci. **28**, 349–350 (1946).

The author derives restrictions on the form of a pair of simultaneous first order differential equations which follow from the assumption that $(y_1+y_2+y_1y_2,\ z_1+z_2+z_1z_2)$ is a solution whenever (y_1, z_1) and (y_2, z_2) are solutions.

P. Franklin (Cambridge, Mass.).

Murnaghan, F. D. The operational calculus. Math. Mag. **21**, 117–138 (1948).

Expository article.

Céressia, Ed. Méthode d'approximations successives pour l'intégration des systèmes linéaires d'équations différentielles. Bull. Soc. Roy. Sci. Liége **16**, 84–93 (1947).

The author gives several different successive approximations procedures for obtaining the integrals of a nonhomogeneous linear system and establishes their convergence.

N. Levinson (Copenhagen).

Germay, R.-H.-J. Sur une généralisation d'un théorème de Lindelöf. Bull. Soc. Roy. Sci. Liége **15**, 442–446 (1946).

The author considers the method of successive approximations for the differential equation $y' = f(x, y)$, when the

0th approximation is not a constant but an arbitrary function. *P. Hartman* (Baltimore, Md.).

Germay, R.-H. Sur une généralisation d'un théorème de Lindelöf pour les systèmes d'équations différentielles de forme normale. Bull. Soc. Roy. Sci. Liége 16, 2–6 (1947).

The author extends the considerations of the paper reviewed above from a single equation to a system of equations. *P. Hartman* (Baltimore, Md.).

Germay, R.-H.-J. Remarque sur une méthode d'approximations successives pour l'intégration des systèmes linéaires d'équations différentielles: extension à des systèmes normaux de forme générale. Bull. Soc. Roy. Sci. Liége 16, 119–125 (1947).

The author modifies the standard method of successive approximations for the system $y_j'=f_j(x, y_1, \cdots, y_n)$, where $j=1, \cdots, n$, so that $y_{i,m}(x)$, the mth approximation for y_j, is defined in terms of $y_{1,m}, \cdots, y_{j-1,m}, y_{j,m-1}, \cdots, y_{n,m-1}$ instead of $y_{1,m-1}, \cdots, y_{n,m-1}$. *P. Hartman.*

Germay, R.-H.-J. Sur une méthode d'approximations successives pour l'intégration des systèmes linéaires d'équations différentielles normales. Bull. Soc. Roy. Sci. Liége 15, 510–513 (1946).

The method referred to in the title is equivalent to replacing the variables $y_1, \cdots, y_n$ in the linear inhomogeneous system $y_i'=\sum_{j=1}^{n}a_{ij}(x)y_j+f_i(x)$ by the variables $u_1, \cdots, u_n$, where $u_i=y_i \exp(-\int^x a_{ii}(t)dt)$, and employing the usual method of successive approximations on the resulting linear system for $u_1, \cdots, u_n$. *P. Hartman.*

de Bruijn, N. G. The logarithmic solutions of linear differential equations. Philos. Mag. (7) 39, 134–140 (1948).

Linear, homogeneous differential equations of the kth order with polynomial coefficients are discussed. Formal solutions of the type $F\sim\sum_{m=-\infty}^{\infty}a(m+t)x^{m+t+r}$, where $a(t)$ is defined and differentiable for all real values of the parameter t, are differentiated with respect to t and these derivatives are also formal solutions of the given equations. If these derivatives have regions of convergence for $t=t_0$, then the formal solutions are actual solutions. In general, such solutions are logarithmic ones except in the case where $a(m)=0$ for all integers m. Applications of the method to Bessel's equation, Legendre's equations and others are given. *W. M. Whyburn* (Lubbock, Tex.).

Butlewski, Z. Sur les intégrales d'un système d'équations différentielles linéaires ordinaires. Studia Math. 10, 40–47 (1948).

The author considers the system $x_j'=\sum a_{jk}(t)x_k+f_j(t)$, where $a_{jk}=b_{jk}+ic_{jk}$, and $i^2=-1$. If the integrals of $|b_{jk}(t)+c_{kj}(t)|$, $|c_{jk}(t)-c_{kj}(t)|$ and $|f_j(t)|$ all exist over (t_0, ∞) then any solution $x_j(t)$ is bounded as $t\to\infty$. [The author states his results for complex t with arg t constant. He does not appear to know that under the more restrictive hypothesis that the $|a_{jk}(t)|$ are integrable, the results are known.] *N. Levinson* (Copenhagen).

Ballieu, Robert. Sur l'unicité de l'intégrale d'une équation différentielle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 725–742 (1947).

The author considers the linear differential equation of nth order, $y^{(n)}+\sum_{k=1}^{n}X_k(x)y^{(n-k)}=0$, where the $X_k(x)$ are continuous functions of a real variable on an interval (or regular analytic functions of a complex variable on a domain). In terms of bounds of the coefficient functions X_k, he determines lower bounds for the diameters of convex regions on which a solution $y(x)\not\equiv 0$ and its derivatives $y^{(1)}, \cdots, y^{(n-1)}$ can all vanish at least once. The result improves that of Fite [Ann. of Math. (2) 18, 214–220 (1917)]. Variations of this problem, such as the determination of the size of the domain on which a solution $y(x)\not\equiv 0$ can vanish n times, are considered. *P. Hartman.*

Levinson, N. A simple second order differential equation with singular motions. Proc. Nat. Acad. Sci. U. S. A. 34, 13–15 (1948).

It is shown that $\ddot{y}+P(y)\dot{y}+y=c \sin t$, where P is a polynomial and c is a constant, has discontinuous recurrent solutions, for appropriate choice of P and c. A similar example was announced by M. L. Cartwright and J. E. Littlewood [J. London Math. Soc. 20, 180–189 (1945); these Rev. 8, 68]; however, the verification of the property asserted for the example of the present paper is much simpler, being based on the explicitly obtainable solutions of a related differential equation. *W. Kaplan.*

McHarg, Elizabeth A. A differential equation. J. London Math. Soc. 22, 83–85 (1947).

The author considers $\ddot{x}+f(x)\dot{x}+g(x)=0$, where $f(x)$ and $g(x)$ are odd functions and positive for $x>0$. If a and b are constants such that $f(x)<ag(x)$ for $0<x<b$ then the equation has a periodic solution satisfying the initial conditions $x=0$, $\dot{x}=v_0$ for every positive $v_0\leqq(1/a, [2G(b)]^{\frac{1}{2}})$, where $G(x)=\int_0^x g(x)dx$. *N. Levinson* (Copenhagen).

Hartman, Philip, and Wintner, Aurel. An oscillation theorem for continuous spectra. Proc. Nat. Acad. Sci. U. S. A. 33, 376–379 (1947).

The "Grenzpunktfall" is considered for the problem $(px')'+(q+\lambda)x=0$, where $p(t)$ and $q(t)$ are continuous for $0\leqq t<\infty$ and $p>0$. In addition there is the boundary condition $x(0)\cos\phi+x'(0)\sin\phi=0$. The authors show that to every λ, $-\infty<\lambda<\infty$, there corresponds at least one boundary condition, that is, one ϕ, such that λ is a point of the spectrum of the Eigenwert problem. *N. Levinson.*

Hartman, Philip, and Wintner, Aurel. The asymptotic arcus variation of solutions of real linear differential equations of second order. Amer. J. Math. 70, 1–10 (1948).

The equation $x''+\omega^2x=0$ is considered, where $\omega(t)$ is positive and continuous for $0\leqq t<\infty$. If $x(t)$ is any real nontrivial solution and $N(t)$ the number of zeros of $x(r)$, $0\leqq r\leqq t$, then $N(t)\sim\pi^{-1}\int_0^t\omega(r)dr$ as $t\to\infty$ if $\omega'(t)=o(\omega^2)$ as $t\to\infty$. The last condition is shown to be best possible in a certain sense. The case $x''-\omega^2x=0$ with the same hypothesis on ω is considered and the conclusion is $x'(t)/x(t)\sim\pm\omega(t)$ as $t\to\infty$. *N. Levinson* (Copenhagen).

Wintner, Aurel. On the location of continuous spectra. Amer. J. Math. 70, 22–30 (1948).

The author considers the spectrum of a differential equation $y''+(\lambda+f(x))y=0$, $0\leqq x<\infty$. If $f(x)$ is continuous and if $-\infty\leqq\limsup_{x\to\infty}f(x)<\infty$ and if for $\lambda=\lambda_0$ all solutions satisfy $y(x)=O(1)$, $y'(x)=O(1)$ as $x\to\infty$ then the equation is of the Grenzpunkt type and λ_0 is a cluster point of the spectrum. A criterion for λ_0 not to be a cluster point is also given. *N. Levinson* (Copenhagen).

Bellman, Richard. On the boundedness of solutions of nonlinear differential and difference equations. Trans. Amer. Math. Soc. **62**, 357–386 (1947).

Let z denote a vector with N components and A a constant matrix. The author considers the system

$$dz/dt=(A+B_1(t))z+B_2(t)dz/dt+f(z, dz/dt, t),$$

where the norms of B_1 and B_2 are small and the first derivatives of $f(z, w, t)$ with respect to z_k and w_k tend to zero as $\|z\|+\|w\|\to 0$, uniformly in t. The system where dz/dt does not appear on the right has been widely treated and the author adopts a familiar integral equation formulation together with a successive approximations technique to treat the case where several or all of the characteristic roots of A have negative real parts. He shows that this technique can be used to study stability in the case where all solutions of $dy/dt=Ay$ are bounded as $t\to+\infty$. Here it is necessary for $\int_0^\infty\|B_i(t)\|dt$ to be small and for the first derivatives of $f(z, w, t)$ with respect to z_k and w_k to satisfy a similar integrability condition. Using the Birkhoff-Kellogg fixed point theorem the author proves corresponding results, some of them new, for $dz/dt=Az+f(z, t)$ under weakened hypotheses of the type $\|f(z, t)\|=o(\|z\|)$ as $\|z\|\to 0$. He also shows how the methods can be used on difference equations.

N. Levinson (Copenhagen).

Ghizzetti, Aldo. Sul comportamento asintotico degli integrali delle equazioni differenziali ordinarie, lineari ed omogenee. Giorn. Mat. Battaglini (4) **1**(77), 5–27 (1947).

This paper arrives simultaneously and independently at several of the results obtained by S. Faedo. [See Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 564–570, 757–764; **3**, 37–43 (1947); these Rev. **9**, 285. The reference to Dunkel given in that review applies here also.]

N. Levinson (Copenhagen).

Plato, G. Über das Abklingen von Schwingungen mit schwacher in beliebiger Weise von der Geschwindigkeit abhängiger Dämpfung. Z. Angew. Math. Mech. **25/27**, 93–94 (1947).

The author considers $\ddot{x}+f(\dot{x})+\omega^2x=0$, where $f(\dot{x})$ is an odd nondecreasing function and moreover $f(\dot{x})$ is taken as very small. In this case he arrives at an approximate formula for the fall in the amplitude of a solution.

N. Levinson (Copenhagen).

Turbovič, I. T. Concerning nonlinear systems with variable parameters. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 203–208 (1948). (Russian)

The author discusses the equation

$$\frac{d}{dt}(\varphi_1(\dot{q})f_1(t))+\varphi_2(\dot{q})f_2(t)+\varphi_3(q)f_3(t)=0$$

which has well-known mechanical or electrical interpretations. "Small parameter" assumptions are replaced by direct assumptions that the logarithmic rates of change of the f_i and the dissipated energy are all small. Assuming a periodic solution the mean rate of change of the energy and the amplitude are actually calculated under the assumption that $\varphi_i(x)/x=|x|^{a_i-1}$, $a_i\geqq 0$. *S. Lefschetz.*

Duncan, W. J. Ignoration of distortional co-ordinates in the theory of stability and control. Coll. Aeronaut. Cranfield. Rep. no. 1, 5 pp. (1946).

Mikusiński, Jan G.-. Sur les intégrales de quelques équations différentielles linéaires. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **1**, 23–34 (1946). (French. Polish summary)

With $L(y)$ designating a differential form

$$L(y)\equiv y^{(n)}+c_1y^{(n-1)}+\cdots+c_ny,$$

in which the coefficients are constants, and with $\varphi(x)$ continuous and nonnegative on (a, b), the paper shows that the solution $y(x)$ of the system

$$L(y)=\varphi(x),\ y(a)=y'(a)=\cdots=y^{(n-1)}(a)=0,$$

is nonnegative on (a, b) if $\eta(x)$ the solution of the system $L(\eta)=0$, $\eta(a)=\eta'(a)=\cdots=\eta^{(n-2)}(a)=0$, $\eta^{(n-1)}(a)=1$, is so. This result is then made the basis for a comparison of the solutions of an equation $y^{(n)}+A(x)y=0$ with those of an equation $y^{(n)}+\mu y=0$, when $0<\mu\leqq A(x)$. Such comparisons are applied to appraise the distances between consecutive zeros of solutions of equations $y^{(n)}+A(x)y=0$, for the cases $n=3$, 4 and 6. *R. E. Langer* (Madison, Wis.).

Mikusiński, Jan G.-. Sur l'équation différentielle $y^{(6)}+y=0$. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **1**, 35–40 (1946). (French. Polish summary)

Each integral $y(x)$ of the differential equation $y^{(6)}+y=0$ for which $y(0)=0$ and $y(x)>0$ in some right-hand neighborhood of $x=0$ is positive in some interval abutting $x=0$ on the right. If the least upper bound of the lengths of these intervals is denoted by λ_6, it is shown in this paper that $15.47<\lambda_6<15.48$. *R. E. Langer* (Madison, Wis.).

Kumar Saha, Ajit. The transmission factors of potential barriers. Proc. Nat. Inst. Sci. India **10**, 373–385 (1944).

The paper is concerned with the "connection formulas" which associate the asymptotic representations of solutions of the one-dimensional wave equation

$$(d^2\chi/dr^2)+(2m/\hbar^2)[E-V-(l(l+1)/r^2)(\hbar^2/2m)]\chi=0,$$

when the coefficient of χ is zero at two values of r and is negative between them. In the main it is an exposition of the reviewer's method for deriving these formulas. The results are compared with those of the less satisfactory methods of Gamow and of Sommerfeld and Bethe.

R. E. Langer (Madison, Wis.).

Meksyn, D. Asymptotic integrals of a fourth order differential equation containing a large parameter. Proc. London Math. Soc. (2) **49**, 436–457 (1947).

The paper is concerned with the differential equation

$$d^4\psi/dy^4+[-2\alpha^2+i\lambda(w-c)]d^2\psi/dy^2+[\alpha^4-i\lambda\alpha^2(w-c)-i\lambda w'']\psi=0,$$

in which α^2 is a constant of moderate magnitude, λ is a large positive parameter, and $w-c$ is an even function of y which has a simple zero at a point y_0 of the fundamental interval $0\leqq y\leqq 1$. Due to this zero the asymptotic solution of the equation relative to λ involves the Stokes phenomenon. In this case three of the four solutions are subject to this phenomenon and the main object of the discussion is the determination of the connection formulas which identify these solutions on opposite sides of the critical point y_0. The method may be outlined as follows. By the changes of variable and parameter

$$\epsilon=[\lambda w'(y_0)]^{-\frac{1}{3}},\quad \eta=(y-y_0)/\epsilon,$$

the equation is transformed into one in which the parameter

is the small quantity ϵ. The assumption of a solution in the form $\psi=f_0+\epsilon f_1+\epsilon^2 f_2+\cdots$ leads to equations

$$d^4f_0/d\eta^4+i\eta d^2f_0/d\eta^2=0,$$
$$d^4f_1/d\eta^4+i\eta d^2f_1/d\eta^2=i\{w''(y_0)/w'(y_0)\}f_0,$$
$$\cdots$$

These equations are solved by infinite definite integrals, and from these expressions the forms of f_0 (or, in one case of $f_0+\epsilon f_1$) are deduced for positive and negative values of η.

R. E. Langer (Madison, Wis.).

Gradšteĭn, I. S. Linear differential equations with small coefficients for the higher derivatives. Doklady Akad. Nauk SSSR (N.S.) **59**, 841–843 (1948). (Russian)

The following theorem is stated, which generalizes previous results by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 391–394 (1946); Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 529–584; these Rev. **8**, 328; **9**, 210] and by Tschen [Compositio Math. **2**, 378–401 (1935)]. Let

$$(1)\qquad \sum_{k=0}^{m}\alpha_k(t,\eta)\frac{d^kX}{dt^k}+\sum_{k=1}^{p}\eta^k\alpha_{m+k}(t,\eta)\frac{d^{m+k}X}{dt^{m+k}}=\alpha_{-1}(t,\eta),$$

where

$$\alpha_k(t,\eta)=\sum_{s=0}^{\infty}\eta^s\alpha_{k,s}(t),\quad k=-1,0,1,\cdots,m,$$

the $\alpha_{k,s}$ being complex functions of t with derivatives of all orders $(0<\eta<\epsilon;\ t_0\leqq t\leqq T)$. Suppose that $\inf|\alpha_{m,0}(t)|>0$, $\inf|\alpha_{m+p}(t)|>0$ and, furthermore, that the equation $\sum_{k=0}^{p}\alpha_{m+k,0}(t)w^k=0$ has no multiple roots, that these roots have negative real parts and that if we denote them by $w_i(t)$ (continuous), the quantities $\Re[w_i(t)-w_j(t)]$ change sign only a finite number of times in (t_0, T). Then, in order that, for $\eta\to 0$, the solution of (1), with the initial values $X^{(k)}(t_0,\eta)=O(\eta^{-k})$, $k=0,\cdots,m+\mu-1$, converge together with the first $m-1$ derivatives towards the solution of the limiting equation with the initial values $x^{(k)}(t_0)$, it is necessary and sufficient that

$$\lim_{\eta\to 0}\left\{\sum_{k=0}^{m-r}\alpha_k(t_0,\eta;r)X^{(k)}(t_0,\eta)+\sum_{k=1}^{p}\eta^k\alpha_{m+k-r}(t_0,\eta;r)X^{(m+k-r)}(t_0,\eta)-\sum_{k=0}^{m-r}\alpha_{k,0}(t_0;r)x^{(k)}(t_0)\right\}=0,\quad r=1,\cdots,m,$$

where the $\alpha_k(t_0,\eta;r)$ are certain explicit linear combinations of the $\alpha_k(t,\eta)$ and its derivatives. No proofs are given.

J. L. Massera (Montevideo).

Bochner, S., and Widder, D. V. A homogeneous differential system of infinite order with nonvanishing solution. Bull. Amer. Math. Soc. **54**, 409–415 (1948).

The differential system $d^{2k-1}[t^kf(t)]/dt^{2k-1}=0$, $f(0+)=f(+\infty)=0$, has no solution other than $f(t)\equiv 0$. Generalizing the differential equation to (*) $\lim_{k\to\infty}L_{k,i}[f]=0$, where $0<t<\infty$ and

$$L_{k,i}[f]=\frac{(-t)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{dt^{2k-1}}[t^kf(t)],$$

and retaining the boundary conditions

$$(**)\qquad f(0+)=f(+\infty)=0,$$

it is seen from the theory of the Stieltjes transform that if the limit (*) holds boundedly, then $f\equiv 0$. However, there do exist nonzero solutions for which (*) holds at each point of $0<t<\infty$. By an exponential change of variable, system (*), (**) is led to the equivalent system (†) $(\sin\pi D)y(x)=0$, $-\infty<x<\infty$, $y(-\infty)=y(+\infty)=0$. (Here D is the operator d/dx.) The following results are obtained: (i) The function $h(x)=e^{-x}(1+e^{-x})^{-2}$, which plays a central rôle, satisfies system (†), as does every even derivative of $h(x)$. (ii) More generally, if $f(w)=\sum_0^\infty a_nw^n$ is an even function of order one and minimal type, then $F(x)=\sum_0^\infty a_nh^{(n)}(x)$ satisfies system (†). (iii) In (ii) the term minimal cannot be replaced by normal, as shown by an example.

I. M. Sheffer (State College, Pa.).

Mendes, M. Sur la forme de l'intégrale d'un système d'équations aux dérivées partielles du premier ordre en involution. J. Math. Pures Appl. (9) **26** (1947), 99–114 (1948).

For a statement of the main results of this paper see C. R. Acad. Sci. Paris **211**, 58–59 (1940); these Rev. **3**, 43.

F. G. Dressel (Durham, N. C.).

Bouligand, Georges. Rôle des intégrales paratingentes en quelques types de problèmes. C. R. Acad. Sci. Paris **226**, 451–453 (1948).

The author is concerned with paratingent integrals (integrals in a certain generalized sense) of a system of partial differential equations of first order. For a special type of system he states a uniqueness theorem which says that two paratingent integrals are identical in case they have in common a sequence of points converging to a point Q_0, with infinite contingent at Q_0. Several related problems are discussed.

H. Samelson (Ann Arbor, Mich.).

Volpato, Mario. Sull'applicazione del metodo degli operatori funzionali all'integrazione di un particolare sistema di equazioni differenziali alle derivate parziali del prim' ordine lineare. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **103**, 37–47 (1944).

Solution by operational methods of the system

$$\partial v(x,t)/\partial t=\partial i(x,t)/\partial x,$$
$$\partial i(x,t)/\partial t=a^2\partial v(x,t)/\partial x,$$

under the conditions

$$v(x,0)=i(x,0)=i(c,t)=0,$$
$$v(0,t)=F(t)+\sum_{n=0}^{n}c_n\partial^{n-m}i(0,t)/\partial t^{n-m}.$$

I. Opatowski (Chicago, Ill.).

Janet, Maurice. Sur la relation entre "équations canoniques" et "problème de calcul des variations" et sur certains systèmes différentiels du second ordre en involution. Bull. Sci. Math. (2) **71**, 62–72 (1947).

Let H be a function of the n variables $y_1,\cdots,y_n$ whose Hessian is of rank $n-p$. Let $\partial H/\partial y_i=Y^i$; then if the variables are so numbered that $|H_{\sigma\tau}|\neq 0$, $\sigma,\tau=p+1,\cdots,n$, the equations $\partial H/\partial y_\sigma=Y^\sigma$ can be solved for y_σ in terms of Y^σ and y_α, $\alpha=1,\cdots,p$. Because the rank is $n-p$ it follows that Y^α are functions of Y^σ alone and so is the function $f=y_iY^i-H$. Conversely if $p+1$ functions $\psi^1,\cdots,\psi^p,f$ of Y^σ are given they can be used to define H as functions of y_i of the required rank. If H contains a number of parameters α^k then under the above conditions y_iY^i-H is a function f of α^k and Y^σ such that $\partial f/\partial Y^\sigma=y_\sigma+y_\beta\partial\psi^\beta/\partial Y^\sigma$ and $\partial f/\partial\alpha^k=-H_{\alpha^k}+y_\beta\partial\psi^\beta/\partial\alpha^k$. If, in particular, α^k are the

$n+1$ variables $x^1, \cdots, x^n, t$, the equations $dx^i/dt=\partial H/\partial y_i$, $dy_i/dt=-\partial H/\partial x^i$ transform into $dx^\sigma/dt=\psi^\sigma(Y^\nu, x, t)$, $dx^\sigma/dt=Y^\nu, dy_i/dt=\partial f/\partial x^i-y_\beta\partial\psi^\beta/\partial x^i$. These considerations are applied to the singular variational problem for the functional f subject to the conditions $dx^\alpha/dt-\psi^\alpha(dx^\sigma/dt, x, t)=0$, the problem being solved by means of Lagrange's multipliers. Another problem the author considers in detail is the system of second-order partial differential equations obtained by setting all $n-p+1$ rowed minors of the Hessian equal to zero. He shows that the general solution depends on $p+1$ functions of $n-p$ variables. *M. S. Knebelman.*

Dufresnoy, Jacques. Sur certains systèmes d'équations différentielles. Bull. Sci. Math. (2) 71, 51–62 (1947).

Soit un système (S) d'équations différentielles linéaires homogènes contenant plus d'inconnues (z) que d'équations. Une remarque simple permet de démontrer le théorème suivant. Soit une famille de solutions caractérisées par leurs valeurs pour $x=\alpha$; si toutes les solutions de la famille vérifient pour $x=\beta$, où $\alpha<\beta$, un même ensemble de r relations indépendantes $C_k{}^i z_i(\beta)=0$, $k=1, \cdots, r$, il existe un système de fonctions $c_k{}^i(x)$ telles que toutes ces solutions satisfassent aux r relations indépendantes $c_k{}^i(x)z_i(x)=0$ pour $\alpha\leqq x\leqq\beta$. Si le système (S) s'écrit $a_i{}^j(x)z_j'+b_i{}^j(x)z_j=0$, $i=1, \cdots, p$; $j=1, \cdots, n$; $p<n$, son adjoint (Σ) sera un système à p inconnues Λ qui s'écrira $(\Lambda^i a_i{}^j)'-\Lambda^i b_i{}^j=0$ et aura au moins r solutions indépendantes; $\|a_i{}^j\|$ étant supposé de rang p, Σ ne peut se réduire à moins de p équations; s'il se réduit effectivement à p équations, il admet p intégrales premières indépendantes, et (S) est dit complètement intégrable; pour qu'il en soit ainsi il faut et il suffit que les relations $a_i{}^j v_j=0$ entraînent les relations $[(a_i{}^j)'-b^j]v_j=0$.

Application au problème de Mayer: soit donné un système (s) de p équations différentielles à n inconnues $f_i(x, y_j, y_j')=0$; parmi toutes les solutions pour lesquelles tous les $y(\alpha)$ sont donnés, ainsi que $y_2, y_3, \cdots, y_n$ pour $x=\beta$, trouver celle pour laquelle $y_1(\beta)$ est extremum. Le théorème obtenu, appliqué au système "aux variations" de (s), montre que le système

$$(\Sigma) \qquad \frac{d}{dx}\left(\Lambda^i\frac{\partial f_i}{\partial y_j'}\right)-\Lambda^i\frac{\partial f_i}{\partial y_j}=0$$

doit présenter au moins une solution. La discussion du système (s, Σ) aux inconnues y, Λ n'est d'ailleurs pas abordée dans le présent article.

Application à l'étude des systèmes de Pfaff: soit donné un système (s) de p équations à n variables $A_i{}^j(y_k)dy_j=0$, $p<n-1$, formons son système "aux variations" (S). Si (s) présente p intégrales premières, il en est de même de (S) pour chaque solution de (s). Réciproquement si (S) présente r intégrales premières pour chaque solution de (s), le système (s) possède aussi r intégrales premières, en particulier si (S) est complètement intégrable, (s) présente aussi p intégrales premières et est dit complètement intégrable. On déduit de là la condition classique pour qu'un système (s) soit complètement intégrable: condition de Frobenius.

M. Janet (Paris).

Cabrera, Nicolas. Perturbation par changement des conditions aux limites. Cahiers de Physique nos. 31–32, 24–62 (1948).

The author considers the problem (1) $H\phi+K^2\phi=0$, where H is a self-adjoint operator and where ϕ satisfies either (a) boundary conditions on a surface which is not a surface of one of the coordinate systems in which (1) is separable or (b) nonuniform, but homogeneous, boundary conditions. He gives an extensive critical review of the previous work in this field. He then proposes a procedure in which the function ϕ is expanded in a series of solution of (1) in which K is left arbitrary. Then K is determined by the direct application of the boundary condition to the series. Explicit formulas are obtained by use of perturbation theory on this sum. An extensive application has been made to the problem of solution of $\nabla^2\phi+K^2\phi=0$ within an ellipse with $\phi=0$ or $\partial\phi/\partial n=0$ on the bounding ellipse. *H. Feshbach.*

Birindelli, Carlo. Sul problema di Neumann-Dini per la striscia piana indefinita. Math. Notae 7, 30–64 (1947).

For the strip $T: 0<y<a$, the author gives in detail a solution of the Neumann-Dini problem of determining a function $u(x, y)$ satisfying $u_{xx}+u_{yy}=\Delta u=f(x, y)$ for a given function $f(x, y)$, and such that the normal derivative assumes given boundary values. The solution is given under the restrictions that $f(x, y)$ satisfies a local Hölder condition in the closure of T and that the prescribed boundary value functions of $\partial u/\partial y$ on $y=0$ and on $y=a$ are continuous functions of x. The behavior of $u(x, y)$ at ∞ is discussed. The solution is developed from the explicit solution of the equation $v_{yy}+\lambda v=0$ in the interval $\epsilon<y<a-\epsilon$, with $v_y=0$ on $y=\epsilon$ and on $y=a-\epsilon$. *E. F. Beckenbach.*

Jensen, H. Højgaard. Some notes on heat-transfer by radiation. Danske Vid. Selsk. Mat.-Fys. Medd. 24, no. 8, 26 pp. (1948).

The loss of energy from a radiating body which is entirely surrounded by another body at a different temperature is investigated here under a variety of quite general conditions. The temperatures of the two bodies are assumed to be steady. Reflected radiation between the bodies is taken into account. The author establishes systems of integral equations which are satisfied by the net radiation, emitted and reflected, of the two bodies. He presents approximate solutions of these for bodies of arbitrary shapes. Formulas are derived for energy losses in several special cases including those of one sphere placed eccentrically about another sphere, a sphere surrounding a flat disk, and a pair of infinite cylinders. A formula by C. Christiansen, dating from 1883, gives the approximate loss of energy by radiation when reflected radiation is present, in some cases. The author points out the situations in which that formula is satisfactory. The assumptions upon which this theory is based are carefully stated and their effects upon the results are thoroughly examined in the paper. *R. V. Churchill.*

Guseĭnov, A. I. On a problem of potential theory. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 103–108 (1948). (Russian)

In this paper there is considered the problem of a stationary temperature distribution in a body for which the coefficient of heat conductivity has different constant values in different parts of the body. The number of these parts is assumed to be finite, and each part constitutes a region bounded by a closed surface of bounded curvature. The problem is reduced to the solution of a system of integral equations which, in contrast to the integral equation corresponding to the ordinary solution of Laplace's equation, has imaginary characteristic values. *H. P. Thielman.*

Kibel', I. A. The heating of a viscous fluid by a rotating disk. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 611–614 (1947). (Russian)

The author considers an infinite plane rotating with constant angular velocity in a viscous incompressible fluid.

Using cylindrical coordinates (r, θ, z) where the z-axis is the axis of rotation, he writes out the partial differential equation satisfied by the temperature at any point in the fluid. He shows that the temperature T can be taken as $T=T_1(z)+T_2(z)r^2$ where T_1 and T_2 are functions of z only, and determines these functions under the boundary conditions

(a) $$\partial T/\partial z=0,\quad dT_1/dz=dT_2/dz=0,\quad z=0,$$
$$T_1(\infty)=T_2(\infty)=0,$$

and (b) $T_1(0)=T_0$, $T_2(0)=0$; $T_1(\infty)=T_2(\infty)=0$.

H. P. Thielman (Ames, Iowa).

Ivancov, G. P. The temperature field around a spherical, cylindrical or pointed crystal growing in a cooling solution. Doklady Akad. Nauk SSSR (N.S.) **58**, 567–569 (1947). (Russian)

The physical problem reduces to the solution of the equation $u_t=a\nabla^2 u$, subject to the boundary condition $u_x^2+u_y^2+u_z^2=f(u)u_t$ at $t=t_0$. The author obtains solutions for the cases mentioned in the title. *R. Bellman.*

Sestini, Giorgio. Sopra un problema ai limiti in un caso non stazionario di propagazione del calore. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 464–477 (1947).

Let R_1 and R_2 be the surfaces of two coaxial cylinders of radii $0<r_1<r_2$ and S_1 the region interior to R_1 while S_2 is the region between R_1 and R_2. The author considers the surface R_1 to have infinitesimal thickness and to be a source of heat $V(t)$. The problem of finding functions $u_1(r, t)$ and $u_2(r, t)$ satisfying

(1) $$\partial^2 u_i/\partial r^2+r^{-1}\partial u_i/\partial r-\partial u_i/\partial t=0$$

in S_i $(i=1, 2)$,

(2) $$(-1)^{i+1}\partial u_i/\partial r+(h_i/\lambda_i)u_i=(h_i/\lambda_i)V(t)$$

on R_1 (h_i, λ_i are coefficients of conductivity), and

$$\partial u_2/\partial r+(h_2/\lambda_2)u_2=0$$

on R_2; $u_i(r, 0)=0$, can be solved by integral equations when $V(t)$ is known. If $V(t)$ is not known, but the total flux of heat across R_1 is measurable, then equations (2) may be replaced by

(2′) $$\lambda_1\partial u_1/\partial r-\lambda_2\partial u_2/\partial r=\lambda H(t)$$

on R_1. The object of the present paper is to show that, if the function $\lambda H(t)$ is given, then the function $V(t)$ can be obtained by solving an integral equation.

F. G. Dressel (Durham, N. C.).

Gatewood, B. E. Note on the thermal stresses in a long circular cylinder of $m+1$ concentric materials. Quart. Appl. Math. **6**, 84–86 (1948).

The general method described by the author in an earlier paper [Philos. Mag. (7) **32**, 282–301 (1941); these Rev. **3**, 248] is applied here to the problem of determining the thermal stresses corresponding to prescribed steady temperatures in composite circular cylinders of infinite length. The cylinders are composed of a solid cylinder and any number of hollow cylinders of different materials, fitted one outside the other. *R. V. Churchill* (Ann Arbor, Mich.).

Rubinštein, L. I. On a question about the propagation of heat in heterogeneous media. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **12**, 27–45 (1948). (Russian)

In this paper there is considered the propagation of heat in a medium which consists of several components. Among the components the heat exchange is assumed to take place according to the law of Henry while within each component the heat propagation follows the law of Fourier. The author states that typical examples of the type of media here considered are soil, molten masses consisting of different kinds of materials at temperatures which correspond to the simultaneous existence of the solid and liquid phases, also structural concrete mixtures. For the particular case of a one-dimensional process in a medium consisting of two components the so-called first and second boundary value problems are completely solved for semi-infinite and bounded intervals. The solution of the problem is obtained by reducing it to the solutions of integral equations of the Volterra type. *H. P. Thielman* (Ames, Iowa).

Reuter, Heinz. Zur Theorie der nächtlichen Abkühlung der bodennahen Schicht und Ausbildung der Bodeninversion. Akad. Wiss. Wien, S.-B. IIa. **155**, 333–358 (1947).

Wilson, A. H. A diffusion problem in which the amount of diffusing substance is finite. I. Philos. Mag. (7) **39**, 48–58 (1948).

Idealizations of problems in the diffusion of dyes into porous materials in commercial dye-baths are treated. The total quantity of dyeing substance in the bath is assumed constant at all times; that is, the amount present in the material plus the amount in solution is constant. Let c denote the concentration of the dye which is freely diffusing in the material and s the concentration of the dye which is deposited in the capillaries of the material. According to the author's first idealization s and c have a constant ratio and c satisfies the equation of diffusion, $c_t=K\nabla^2 c$. The problem of determining c, a function of x and t, in a strip of finite thickness placed in the center of a dye-bath of finite width is solved first. The boundary value problem is the same as the one satisfied by the temperatures in a slab immersed in a finite quantity of well-stirred liquid. The author solves it by introducing as a new unknown function an integral of c with respect to x, so that a Sturm-Liouville expansion can be used. A simple formula is written for the total amount of dye present in the strip at any time. The corresponding problem in the concentrations $c(r, t)$ in a cylindrical substance is solved. Without such a severe idealization as that used above the concentrations satisfy the equations $c_t=k\nabla^2 c-s_t$, where $s_t=\lambda c-\mu s$ and the coefficients are constants. The boundary value problem under these conditions is solved for the cylindrical substance, using Laplace transforms. *R. V. Churchill* (Ann Arbor, Mich.).

Crank, J. A diffusion problem in which the amount of diffusing substance is finite. II. Diffusion with non-linear absorption. Philos. Mag. (7) **39**, 140–149 (1948).

[Cf. the preceding review.] A porous circular cylinder of infinite length is placed in a dye solution. Let a denote the radius of the cylinder and A the area of the cross section of the dye-bath exclusive of the space occupied by the cylinder. The concentration of dye in the solution in the bath itself is always uniform, initially C_0. Let $C(r, t)$ denote the concentration of the dye in the solution inside the cylinder, where r is the cylindrical coordinate, and let $S(r, t)$ denote the concentration of the dye that has been deposited in the capillaries of the cylinder, where S is a known nonlinear function of C. The function C satisfies the partial differential equation $C_t=(rC_r)_r k/r-S_t$, the boundary condition $AC(a, t)+2\pi\int_0^a r(C+S)dr=AC_0$ and the initial condition

$C(r, 0)=0$. After introducing dimensionless parameters the author finds approximate solutions of this boundary value problem by the method of finite differences for several cases in which S is proportional to C^m, $0 \leqq m \leqq 1$.

R. V. Churchill (Ann Arbor, Mich.).

Bose, S. N., and Kar, S. C. The complete solution of the equation: $\nabla^2\phi - \frac{\partial^2\phi}{c^2\partial t^2} - k^2\phi = -4\pi\rho(xyzt)$. Proc. Nat. Inst. Sci. India 7, 93–102 (1941).

The scalar potential and each component of the vector potential of the field due to a moving mesotron satisfy the equation $\nabla^2\phi - c^{-2}\partial^2\phi/\partial t^2 - k^2\phi = 0$. This equation has a solution χ of the form $kY_1(ks)/s$, where $s^2 = c^2(\theta-t)^2 - (\xi-x)^2 - (\eta-y)^2 - (\zeta-z)^2$, which plays the same role as the solution $1/s^2$ of the equation of wave motions. Bhabha's expressions for these potentials (which naturally reduce to the formulae of Liénard and Wiechert for the case $k=0$) are obtained here from the solution χ by a contour integration method which Bateman [The Mathematical Analysis of Electrical and Optical Wave Motion, Cambridge University Press, 1915, p. 115] attributed to A. W. Conway, for the case $k=0$; the present authors call this the method of Herglotz and Sommerfeld.

For the nonhomogeneous equation of the title, the authors obtain a formula similar to Kirchhoff's solution of the nonhomogeneous wave equation. Two proofs of this are given, one being by means of contour integration, the other a straightforward generalisation of the Beltrami-Love proof of Kirchhoff's formula. *E. T. Copson* (Dundee).

Malmheden, Harry. A class of hyperbolic systems of linear differential equations. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] 8, 1–116 (1947).

The aim of this thesis is to extend Riesz's method of solving a hyperbolic partial differential equation of the second order by means of integrals of fractional order, to cover the case of a system of first order equations

$$\sum_{i,r} A^i_{rs}\partial u^r/\partial x^i + \sum_r B_{rs}u^r = f_s, \qquad s=1, \cdots, n,$$

involving m independent variables x^i and n dependent variables u^r. It is convenient to write this in the matrix form

$$\sum_i \mathbf{A}^i\partial\mathbf{u}/\partial x^i + \mathbf{Bu} = \mathbf{f}, \tag{1}$$

where $\mathbf{A}^i$ and $\mathbf{B}$ are square matrices and $\mathbf{u}$ a column matrix.

After discussing the characteristics of this system, the idea of a multiplicator is introduced. In the case when the coefficients are constants and $\mathbf{B}=0$, a multiplicator is a differential operator which separates the dependent variables, i.e., which turns the system into one of the form $L_r u^r = F^r$, where L_r is a certain differential operator. The simplest operator of this type is the adjugate of the matrix $\sum_i \mathbf{A}^i\partial/\partial x^i$, and in this case the operator L_r is the same for all r and is of order n. It can, however, happen that the system has a linear multiplicator of the form $\sum \mathbf{C}^i\partial/\partial x^i$. A necessary and sufficient condition for this is that n be even, and that the determinant of the matrix $\sum \mathbf{A}^i p_i$ be equal to $K^{n/2}$, where K is an irreducible quadratic form, $K=\sum_{i,k} g^{ik}p_i p_k$, and that all the first minors be divisible for $K^{(n-2)/2}$. The multiplicator then turns the system into $\sum g^{ik}\partial^2\mathbf{u}/\partial x^i\partial x^k = \mathbf{F}$.

If the coefficients are variable, it is not possible to obtain complete separation of the variables, but only separation of the highest derivatives. Thus, if a linear multiplicator exists, the system is reducible to

$$\vec{\Delta}\mathbf{u} = \frac{1}{\sqrt{g}}\sum \frac{\partial}{\partial x^i}\left(\sqrt{g}\cdot g^{ik}\frac{\partial \mathbf{u}}{\partial x^k}\right) + \mathbf{h}^i\frac{\partial \mathbf{u}}{\partial x^i} + \mathbf{lu} = \mathbf{F}. \tag{2}$$

The characteristic conoid $G=0$ is determined for this equation without introducing the idea of null geodesics.

Instead of using the elementary solution which is infinite on $G=0$, the Riesz parameter kernel is introduced. This is a square matrix $\mathbf{U}^{(\alpha)}$ which depends on a parameter α, which satisfies $\vec{\Delta}\mathbf{U}^{(\alpha)} = \mathbf{U}^{(\alpha-1)}$ and vanishes on $G=0$ if the real part of α is large enough. The problem of Cauchy is then solved for the system (2) by the Riesz technique of analytical continuation with respect to α, and from this the solution of the same problem for (1) is deduced.

The thesis concludes with an application to the second order system (2) for the case when the variables u^r are the components of a vector in the space of metric $ds^2 = \sum g_{ik}dx^i dx^k$.

E. T. Copson (Dundee).

Gårding, Lars. Équations différentielles linéaires hyperboliques à coefficients constants. C. R. Acad. Sci. Paris 226, 539–541 (1948).

This note is concerned with a partial differential equation (*) $q(\partial/\partial x)u = f$ which has constant coefficients and degree m. Here $q(\zeta) \equiv q(\zeta_1, \cdots, \zeta_n)$ is a polynomial which is not a function of fewer than n linear combinations of the variables ζ_k. If one writes $q=p+r$, where p is homogeneous of degree m and r is the sum of the terms of lower degree, q is said to be hyperbolic with respect to the real vector $\vec{\xi}=(\xi_1, \cdots, \xi_n)$ if $p(\vec{\xi})\neq 0$ and if there exists a real number t_0 such that $q(t\xi+i\eta)\neq 0$ for all real vectors η and for all $t>t_0$. This implies that $p(t\xi+\eta)$ vanishes for m real values of t, say $t=u_k(\xi, \eta)$, where $k=1, 2, \cdots, m$. If one denotes by Γ the set of all ξ such that $u_k(\xi, \xi)>0$, then Γ is the interior of a convex cone containing $\vec{\xi}$, and q is hyperbolic with respect to every ξ of Γ. If $\xi - t_0\vec{\xi}$ belongs to Γ and if the real part of α is positive, $q(\xi+i\eta)^{-\alpha}$ is the Fourier-Laplace transform of a continuous function $Q(\alpha, x)$ which vanishes outside a certain convex cone. This function $Q(\alpha, x)$ plays the same part in the theory of the equation (*) as Riesz's function $\Gamma^{(\alpha-n)/2}$ does in the theory of the equation of spherical waves. In fact, if one writes $I^\alpha f(x) = \int Q(\alpha, x-y)f(y)dy$, one has $q(\partial/\partial x)I^{\alpha+1}f(x) = I^\alpha f(x)$. Then $I^\alpha f(x)$ is an analytic function of α, regular when the real part of α is positive, and $I^0 f(x) = f(x)$. From this, the solution of the problem of Cauchy for (*) follows at once. *E. T. Copson* (Dundee).

Beck, Guido. Solutions of Dirac's equations in hypercomplex form. Anais Acad. Brasil. Ci. 19, 321–332 (1947).

Making use of polar and cylindrical coordinates, the author obtains solutions of Dirac's equations for the cases of motion in a Coulomb field and in a homogeneous magnetic field, respectively. He regards his method, involving the use of hypercomplex functions, as being new. However, the reviewer has been unable to perceive any essential novelty in the method or in the results. There are many unexplained symbols, or perhaps typographical errors, and in general the entire paper is far from being lucid.

L. A. MacColl (New York, N. Y.).

Piskunov, N. S. On the characteristic problem for equations of ultrahyperbolic type. Doklady Akad. Nauk SSSR (N.S.) 59, 439–442 (1948). (Russian)

The ultrahyperbolic equation $u_{xx}+u_{yy}-u_{zz}-u_{tt}=0$ is considered. The values of u are assumed to be given on th

characteristic cone $R-r=0$, where $R^2=x^2+y^2$, $r^2=z^2+t^2$. Under the assumption that the boundary values admit certain developments in Fourier and power series a solution is constructed for $R\leqq r$. Sufficient conditions for uniqueness of the solution are stated. *P. A. Lagerstrom.*

Bureau, Florent. Sur la solution élémentaire d'une équation linéaire aux dérivées partielles d'ordre quatre et à trois variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 473–484 (1947).

Discussion of the elementary solution for an elliptic equation of the form

$$\left(a_1\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}+a_3\frac{\partial^2}{\partial z^2}\right)\left(b_1\frac{\partial^2}{\partial x^2}+b_2\frac{\partial^2}{\partial y^2}+b_3\frac{\partial^2}{\partial z^2}\right)u=0$$

(a_i, b_i positive constants). The author identifies his elementary solution with one used by Brillouin in the study of an electromagnetic field in a biaxial medium. *F. John.*

Bureau, Florent. Les solutions élémentaires des équations linéaires aux dérivées partielles totalement hyperboliques d'ordre plus grand que deux et à un nombre impair de variables indépendantes. C. R. Acad. Sci. Paris **225**, 852–854 (1947).

The author announces an explicit solution of the Cauchy problem for a totally hyperbolic differential equation of the form

$$f(\partial/\partial x, \partial/\partial y, \partial/\partial t)u=g(x, y, t),$$

where $f(\alpha, \beta, \gamma)$ is a form of degree n with constant coefficients, which is even in γ. The solution is constructed with the help of an elementary solution of the form

$$v(x, y, t)=i\pi\sum\epsilon_\nu\int_{(\alpha_0\beta_0)}^{(\alpha\beta\nu)}(\alpha x+\beta y+t)^{n-1}f_\beta^{-1}d\alpha,$$

where the sum is extended over all complex intersections of the line $\alpha x+\beta y+t=0$ with the curve $f(\alpha, \beta, 1)=0$, ϵ_ν stands for $\operatorname{sgn}\Re(\beta_\nu/i\alpha_\nu)$ and (α_0, β_0) is an arbitrary point. The same method is said to work for any odd number of independent variables. It seems to be similar to the one used by Herglotz [Abh. Math. Sem. Hamburgischen Univ. **6**, 189–197 (1928)]. *F. John* (New York, N. Y.).

Hadamard, Jacques. Observation sur la note précédente. C. R. Acad. Sci. Paris **225**, 854 (1947).

Cf. the preceding review.

Bureau, Florent. Les solutions élémentaires des équations linéaires aux dérivées partielles totalement hyperboliques d'ordre plus grand que deux et à quatre variables indépendantes. C. R. Acad. Sci. Paris **226**, 150–152 (1948).

A solution of the same type as the one discussed in the second preceding review is given for totally hyperbolic equations in four independent variables. *F. John.*

Cinquini-Cibrario, Maria. Un teorema di esistenza e di unicità per un sistema di equazioni alle derivate parziali. Ann. Mat. Pura Appl. (4) **24**, 157–175 (1945).

The Cauchy problem for a hyperbolic equation of order n

$$(1)\qquad F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \cdots, \frac{\partial^n z}{\partial y^n}\right)=0$$

for a function $z(x, y)$ formed the subject of a paper by K. O. Friedrichs and H. Lewy [Math. Ann. **99**, 200–221 (1928)]. There the problem was reduced to a system of "canonical" equations holding along the characteristics of (1). However the proof for the existence of a solution given in that paper was incomplete, as it did not take into account that some of the coefficients in the canonical system depended on the derivatives of the unknown functions.

The present paper gives existence proofs for solutions of a "canonical" system of equations, which is general enough to include the relations derived from (1) along the characteristic lines, and hence completes the proof for the existence of solutions of the Cauchy problem. The canonical system used by the author can be written as a system of integral equations for functions $u_\nu(\lambda, \mu)$ along paths which are solutions of differential equations depending on the u_ν. These integral equations are of a rather unusual type, but can be put into a form where solution by iteration is possible. Considerable difficulties are overcome by the author in proving that the iteration process converges and furnishes functions with the required number of derivatives.

[Similar results have been announced by K. O. Friedrichs, Bull. Amer. Math. Soc. **53**, 910 (1947).] *F. John.*

Salehov, G. S. On the problem of Cauchy for a class of partial differential equations in the domain of arbitrarily smooth functions. Doklady Akad. Nauk SSSR (N.S.) **59**, 857–859 (1948). (Russian)

The author discusses the nature of the solutions of the partial differential equation $\partial^p z/\partial t^p-\epsilon t^m\partial^q z/\partial x^q=0$, $\epsilon=\pm 1$, where m is a nonnegative integer and $|t|\leqq R$, $a\leqq x\leqq b$. Calling p/q the weight of the equation, several theorems are given showing the relation between the weight of the equation and the analytic properties of the solutions. No proofs are given. *R. Bellman* (Stanford University, Calif.).

Integral Equations

Horn, Alfred. The asymptotic behavior of solutions of systems of Volterra integral equations. Trans. Amer. Math. Soc. **63**, 144–174 (1948).

The author considers the system of Volterra integral equations

$$(1)\quad u_i(x, y, \lambda)=\sum_{j=1}^{n}\lambda\int_y^x k_{ij}(x, t, \lambda)u_j(x, y, \lambda)dt+f_i(x, y, \lambda),$$

where k_{ij} and f_i can be developed in certain regions of the complex λ-plane in series of the form

$$(2)\qquad \begin{aligned} k_{ij}(x, y, \lambda)&=k_{ij}^0(x, y)+k_{ij}^1(x, y)/\lambda+\cdots,\\ f_i(x, y, \lambda)&=f_i^0(x, y)+f_i^1(x, y)/\lambda+\cdots.\end{aligned}$$

He investigates the asymptotic behavior of the solutions of the system (1), on the assumption that the characteristic roots of the matrix $(k_{ij}^0(x, y))$ are distinct and different from zero for every x in the closed interval (α, β) in which x and y vary. First the author derives the formal series solutions of (1), investigating their asymptotic character. In these series the coefficients are determined by solving first-order linear differential equations and systems of Volterra equations of the first kind. Other particular cases are considered. The results are applied to give the asymptotic behavior of solutions of integro-differential equations of the form

$$D^{(n-1)}(x, y)+\sum_{i=1}^{n-1}\lambda^i\alpha_i(x, y)D^{(n-1-i)}(x, \lambda)+\lambda^n\int_a^x\Gamma(x, t, \lambda)D(t, \lambda)dt=\lambda^n\Phi(x, y),$$

where the given functions have asymptotic developments of the form (2). Finally, the author investigates the convergence of the solution of the equation

$$D(x)=\lambda\int_a^x \Gamma(x,t)D(t)dt+\lambda\Phi(x)$$

to the solution of the equation

$$\int_a^x \Gamma(x,t)D(t)dt+\Phi(x)=0.$$

C. Miranda (Naples).

Sarmanov, O. V. On the rectification of asymmetrical correlation. Doklady Akad. Nauk SSSR (N.S.) **59**, 861–863 (1948). (Russian)

The paper deals with the problem of the linearizations of the lines of regression. Let $F(x, y)$ be a nonsymmetrical distribution function of two variables which determines a correlation in the whole plane. If the correlation is curvilinear, i.e., if the lines of regression $f_1(x)$, of y on x, and $f_2(y)$, of x on y, are nonlinear functions, then there exists the problem of finding such transformations of coordinates $X=\varphi(x)$, $Y=\psi(y)$ that the correlation between X and Y is linear. If such transformations exist the correlation is said to be rectifiable. The author states that the rectification of a correlation is equivalent to finding a pair of monotone solutions of the system of the so-called integral equations of correlation,

$$\varphi(x)=\lambda\int_{-\infty}^{\infty}\psi(y)\{F(x,y)/p(x)\}dy,$$
$$\psi(y)=\lambda\int_{-\infty}^{\infty}\varphi(x)\{F(x,y)/P(y)\}dx, \tag{1}$$

where $p(x)$ and $P(y)$ are the marginal densities of x and y, respectively. The system (1) is shown to be equivalent to another system of integral equations involving symmetrical distribution functions, which were treated by the author in an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 773–776 (1946); these Rev. **8**, 467]. The author extends some of the results of that paper on symmetric distribution functions to the nonsymmetric case and shows that if the conditional distribution functions of y for a given x, and of x for a given y, satisfy certain conditions, then the system (1) will have monotone solutions $\varphi_1(x)$ and $\psi_1(y)$ corresponding to the first characteristic number λ_1 whose absolute value is greater than one. The transformation $X=\varphi_1(x)$, $Y=\psi_1(y)$ will yield a linear correlation with a coefficient of correlation equal to $1/\lambda_1$. *H. P. Thielman* (Ames, Iowa).

Sarmanov, O. V. On the order of magnitude of a line of regression. Doklady Akad. Nauk SSSR (N.S.) **59**, 1061–1064 (1948). (Russian)

Let $F(x, y)$ be the density of the distribution of two variables which determines a correlation in the whole plane, and satisfies the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F^2(x,y)}{p(x)P(y)}dxdy=K<\infty, \tag{1}$$

where $p(x)$ and $P(y)$ are the marginal densities of x and y, respectively. On the basis of the results of an earlier paper [reference in the preceding review] the author proves the following theorems. There exists no symmetrical correlation which satisfies condition (1) and which is such that the lines of regression $\varphi(x)$, of y on x, satisfy the condition $|\varphi(x)|\geqq c^{-1}|x|^{1+\alpha}$, $c>0$, $\alpha>0$, for $|x|>A$, while at the same time $\liminf \varphi_1(x)>0$, where

$$\varphi_1(x)=\int_{-\infty}^{\infty}|y|\{F(x,y)/p(x)\}dy.$$

There exists no nonsymmetrical correlation satisfying condition (1) which is such that the lines of regression $\varphi(x)$, of y on x; $\psi(y)$, of x on y, satisfy the conditions

$$|\varphi(x)|\geqq c^{-1}|x|^{1+\alpha},\ c>0,\ \alpha>0, \qquad |x|>A,$$
$$|\psi(y)|\geqq c_1^{-1}|y|^{1/(1+\alpha)+\beta},\ c_1>0,\ \beta>0, \qquad |y|>B,$$

while at the same time $\liminf \varphi_1(x)>0$, $\liminf \psi_1(y)>0$, where $\psi_1(y)$ has an analogous meaning to that given for $\varphi_1(x)$ above. The following results are stated as corollaries. If for any correlation which satisfies condition (1) the lines of regression are polynomials of positive degree, they must be linear. If the lines of regression $\varphi(x)$, $\psi(y)$ are algebraic curves of degrees $\delta>0$, $\delta_1>0$, then $\delta\delta_1\leqq 1$.

H. P. Thielman (Ames, Iowa).

Trjitzinsky, W. J. Singular integral equations with complex valued kernels. Ann. Mat. Pura Appl. (4) **25**, 197–254 (1946).

The author considers integral equations involving singular kernels $K(x, y)=K'(x, y)+iK''(x, y)$, where K', K'' are real and are limits of regular kernels $K_n=K_n'+iK_n''$ with real characteristic functions, giving a general expression for a kernel of the latter type in terms of the characteristic values and functions of its real part. The paper then deals with the spectral development of such kernels by methods established, as the author points out, by T. Carleman [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923]. The results of this section include theorems on the spectral representation of iterated symmetrical kernels of the same type.

It is shown that, when $K(x, y)$ is symmetric and satisfies certain mean square continuity conditions, the general solution of the equation

$$\phi(x)=\lambda\int_0^1 K(x,y)\phi(y)dy+f(x)$$

is a limit of solutions of the approximating equation with regular kernels $K_n(x, y)$; and the general form of the solution is found in terms of the spectral representation of $K(x, y)$. This generalizes similar results of Carleman in that the characteristic values of the approximate kernels may be complex. Further extensions of the theory are indicated.

H. R. Pitt (Belfast).

Šerman, D. I. On a case in the theory of singular equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 647–650 (1948). (Russian)

The author studies the equation

$$A(t_0)w(t_0)+(\pi i)^{-1}B(t_0)\int(t-t_0)^{-1}w(t)dt+\int w(t)K(t,t_0)dt=f(t_0), \tag{1}$$

where integration is along a simple "smooth" contour L; t, t_0 are on L; the coefficients are essentially of a Hölder class on L. In previous literature (1) has been transformed into a regular Fredholm equation of the second kind, predominantly under the supposition that (2) $A^2-B^2\neq 0$ on L. The author gives a new method for effecting such a trans-

formation when (2) does not hold. The case actually carried out is the one when $A-B$ has just one simple zero on L. The question of equivalence of the resulting Fredholm equation and of (1) is examined. The results obtained can be extended to systems of equations analogous to (1).

W. J. Trjitzinsky (Urbana, Ill.).

Rapoport, I. M. On a class of singular integral equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 1403–1406 (1948). (Russian)

The equation considered is (1) $f(x)+\int_0^\infty f(t)k(x-t)dt=g(x)$. The method of N. Wiener and E. Hopf [S.-B. Preuss. Akad. Wiss. Math.-Phys. Kl. **1931**, 696–706] is applicable only when $k(x)=O(e^{-\lambda|x|})$, $\lambda>0$. The author solves the problem of existence and uniqueness of the solution of (1) in the class L_2, for g in L_2; also, the solution of (1) is given by quadratures when $k(x)$ is L_2 and $K(x)(2\pi)^{\frac{1}{2}}=\int_{-\infty}^{\infty}k(t)e^{ixt}dt$ satisfies: (2) $K(x)$ is Hölder α $(\alpha>0)$ for $-\infty<x<\infty$; $K(x)=O(|x|^{-\beta})$ $(\beta>0$, for $|x|\to\infty)$; $K(x)\not\equiv-(2\pi)^{-\frac{1}{2}}(-\infty<x<\infty)$. The author's method is based on the following. In order that $\psi=\psi_1+i\psi_2$ (ψ in L_2) should represent on the axis of reals the boundary values of a function analytic in the upper (lower) half-plane and vanishing at ∞, it is necessary and sufficient that $\int_{-\infty}^{\infty}\psi(t)e^{ixt}dt$ be zero on the positive (negative) real semiaxis.

W. J. Trjitzinsky (Urbana, Ill.).

Mihlin, S. G. On the integral equation of F. Tricomi. Doklady Akad. Nauk SSSR (N.S.) **59**, 1053–1056 (1948). (Russian)

In his study of equations of mixed elliptic-hyperbolic type F. Tricomi [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 133–247 (1923)] was led to the consideration of the equation

$$(1)\quad v(x)-\lambda\int_0^1 (x^{-1}y)^{\frac{1}{2}}\{(y-x)^{-1}-(x+y-2xy)^{-1}\}v(y)dy=f(x).$$

The existing theory of singular (in the sense of principal values) integral equations is not applicable since the "non-singular" component $(x+y-2xy)^{-1}$ is not integrable over $(0\leqq x\leqq 1, 0\leqq y\leqq 1)$. Tricomi gave a long and complicated solution. The author lets $x^{\frac{1}{2}}v(x)=\varphi(x)$, $x^{\frac{1}{2}}f(x)=g(x)$, so that (1) becomes

$$(2)\quad \varphi(x)-\lambda\int_0^1\{(y-x)^{-1}-(x+y-2xy)^{-1}\}\varphi(y)dy=g(x);$$

$g(x)$ is assumed of a Hölder class; solutions φ are sought in the class of functions such that: (a) φ is of a Hölder class on every interval $0<a\leqq x\leqq b<1$; (b) $\varphi(x)\log x$ and $\varphi(x)\log(1-x)$ are summable on (0, 1). The author applies to (2) successfully and concisely the method of T. Carleman [Ark. Mat. Astr. Fys. **16**, no. 26 (1922)].

W. J. Trjitzinsky (Urbana, Ill.).

Guseĭnov, A. I. On a class of nonlinear singular integral equations. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 193–212 (1948). (Russian)

The author studies the nonlinear singular integral equation (1) $u(x)=\lambda\int_a^b(s-x)^{-1}K(x,s,u(s))ds$ (small parameter λ; integration in the sense of Cauchy principal values). A space H is introduced for whose elements one has: $|u(x)|\geqq k|x-a|^{-\alpha}|x-b|^{-\beta}$; $|u(x+\Delta x)-u(x)|\leqq kpq$, where $p=|x-a|^{-\alpha}|x+\Delta x-b|^{-\beta}$, $q=|\Delta x|^{\delta}|x-a|^{-\delta}|x+\Delta x-b|^{-\delta}$ $(k>1; 0<\alpha,\beta,\alpha+\delta,\beta+\delta<1)$, $|\Delta x|<\min(\frac{1}{4}|x-a|,\frac{1}{4}|x-b|)$. The space H is convex. The metric used in H is $\rho(u,v)=\max|u-v|\cdot|x-a|^{\alpha+\delta}\cdot|x-b|^{\beta+\delta}$. Existence of a solution of (1) (in H) is based on a fixed point principle of J. Schauder and is established when

$$(2)\quad |K(x,s,u(s))|<A|u|+B,$$
$$|K(x+\Delta x,s+\Delta s,u(s+\Delta s))-K(x,s,u(s))|<\nu,$$

where

$$\nu=A_1|\Delta x|^{\delta_1}+A_2|\Delta s|^{\delta}+A_3|u(s+\Delta s)-u(s)|$$

$(0<\delta<\delta_1<1)$. If K satisfies (2) and $|K_u'(x,s,u(s))|<l$, $|K_u'(x+\Delta x,s+\Delta s,u(s+\Delta s))-K_u'(x,s,u(s))|<\tau_1+\tau_2$, where $\tau_1=a_1|\Delta x|^{\delta_1}+a_2|\Delta s|^{\delta}$ and

$$\tau_2=a_3|u(s+\Delta s)-u(s)|\,|s-a|^{\alpha}\cdot|s+\Delta s-b|^{\beta},$$

then the solution, in H, of (1) is unique (for λ small) and can be obtained by successive approximations.

W. J. Trjitzinsky (Urbana, Ill.).

Zagadskiĭ, D. M. An analogue of Newton's method for nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 1041–1044 (1948). (Russian)

The method of successive approximations is applied to the equation (1) $\varphi(x)=\int_a^b K(x,y,\varphi(y))dy$. Assumptions (i), (ii), (iii) are made as follows: (i) $K(x,y,z)$, $K_z'(x,y,z)$ are continuous in x, y and z $(a\leqq x, y\leqq b; |z|\leqq L)$; $K_{zz}''(x,y,z)$ exists and is bounded; (ii) the resolvent $T_0(x,y)$ of $K_\varphi'(x,y,\varphi_0(y))$ exists, $\varphi_0(x)$ being continuous and bounded, while $\limsup_x\int|T_0(x,y)|dy=B<\infty$; (iii) $\epsilon_0\leqq.1\delta^{-1}(1+B)^{-1}$, where $\delta=P(1+B)(b-a)$. With $\varphi_0(x)$ continuous, the author considers the equation

$$(2)\quad \varphi(x)=\int K(x,y,\varphi_{k-1}(y))dy+\int K_\varphi'(x,y,\varphi_{k-1}(y))\{\varphi(y)-\varphi_{k-1}(y)\}dy,\qquad k=1,2,\cdots.$$

Under (i), (ii), (iii), (2) has a unique continuous solution $\varphi_k(x)$ so that $|\varphi_k-\varphi_0|\leqq h$, where

$$5\delta h=1-(1+10\delta\epsilon_0(1+B))^{\frac{1}{2}},$$

$k=1,2,\cdots$. Moreover, under (i), (ii), (iii), (1) has a continuous solution $\varphi^*(x)$ in the domain $|\varphi(x)|\leqq L$; φ^* is the limit of a uniformly convergent sequence $\varphi_k(x)$; in the domain $a\leqq x\leqq b$, $|\varphi(x)-\varphi_0(x)|\leqq h$, this solution is unique.

W. J. Trjitzinsky (Urbana, Ill.).

Robinson, Lawrence Baylor. Application of the Laplace transform in the solution of linear integral equations. J. Appl. Phys. **19**, 237–241 (1948).

The author discusses integral equations of the Abel and Volterra type with Faltung kernels and their solution by the unilateral Laplace transform. It is stated that "the integral equations of greatest importance to the physicists are those known as Abel's equations and Volterra's equation of the second kind," thereby overlooking the recent work on the Wiener-Hopf integral equation. *A. E. Heins.*

Cârstoiu, Ion. Sur la détermination du coefficient d'hérédité d'une corde élastique, par le calcul symbolique. C. R. Acad. Sci. Paris **226**, 1154–1155 (1948).

As a direct consequence of the convolution property of the Laplace transform, called the Borel theorem in operational calculus, linear integral or integrodifferential equations of the convolution type transform into algebraic equations in the transform of the unknown function. The author trans-

forms the equation $kf''(t)+f(t)+\int_0^t f(\tau)\psi(t-\tau)d\tau=0$ and discusses the determination of ψ when f is given. The function f represents the displacement of a mass attached to the end of an elastic cord and the function ψ denotes a fatigue coefficient for the restoring force.

R. V. Churchill (Ann Arbor, Mich.).

Mark, J. Carson. Milne's problem for anisotropic scattering. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-26**, i+35 pp. (1943).

In this paper the problem of the diffusion of neutrons in a semi-infinite plane-parallel atmosphere with a constant net flux and scattering (without change of energy) in accordance with a phase function of the form $p(\cos\Theta)=1+p_1\cos\Theta+p_2\cos^2\Theta$ is considered. The method used is a generalization of the one described by G. Placzek and W. Seidel [Physical Rev. (2) **72**, 550–555 (1947); these Rev. **9**, 147]. [It may be noted that the solution to the same problem has been obtained by different methods by S. Chandrasekhar, Astrophys. J. **105**, 164–203 (1947); these Rev. **8**, 467; see particularly sections IV and V; see also S. Chandrasekhar and F. H. Breen, Astrophys. J. **105**, 435–440 (1947); these Rev. **9**, 40, and the paper reviewed below, where tables of solutions will be found.]

S. Chandrasekhar (Williams Bay, Wis.).

Chandrasekhar, S., and Breen, Frances H. On the radiative equilibrium of a stellar atmosphere. XIX. Astrophys. J. **106**, 143–144 (1947).

[Paper XVIII appeared in the same J. **105**, 461–470 (1947); these Rev. **9**, 41.] The present paper extends the tabulation of H-functions begun in paper XVI [same J. **105**, 435–440 (1947); these Rev. **9**, 40] to the case of diffuse reflection by a semi-infinite isotropically scattering atmosphere. Here the characteristic function $\Psi(\mu)$ is a constant equal to $\frac{1}{2}\omega_0$, where $\omega_0(\leqq 1)$ is the albedo. *F. Smithies.*

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XX. Astrophys. J. **106**, 145–151 (1947).

[Cf. the preceding review.] The author solves the equation of transfer in the case when the combined effects of monochromatic line scattering and general continuous absorption are taken into account, under the assumptions (i) that the ratio of the line and continuous absorption coefficients is constant throughout the atmosphere and (ii) that the Planck intensity is a linear function of the optical depth (in terms of the combined absorption coefficients). The solution is in terms of the H-functions tabulated in paper XIX [see the preceding review], and their first and second moments, which are tabulated here.

F. Smithies (Cambridge, England).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XXI. Astrophys. J. **106**, 152–216 (1947).

[Cf. the two preceding reviews.] In this paper the author begins the consideration of the theory of diffuse reflection and transmission of radiation by a plane parallel atmosphere of finite optical thickness. He shows that the methods introduced in his earlier papers [see especially paper XIV, same J. **105**, 164–203 (1947); these Rev. **8**, 467] for the case of a semi-infinite plane parallel atmosphere can be adapted to the present case. Approximate solutions are given in the present paper and the discussion of the exact solutions is promised for a later paper.

The author incidentally discusses the problem of finding a polynomial $F(x)$ of degree n such that

$$F(x_\alpha)=\lambda_\alpha F(-x_\alpha), \qquad \alpha=1, \cdots, n,$$

where the x_α are n distinct values of the argument and the λ_α are n distinct numbers. He shows that the polynomial $F(x)$ is unique, apart from a multiplicative constant, and gives an explicit formula for it in terms of the x_α and λ_α. He also gives the general form of polynomials of degree $n+m$, where $m>0$, satisfying the same conditions.

F. Smithies (Cambridge, England).

Berlin, T. H. Note on the Milne problem for a sphere. Physical Rev. (2) **73**, 437–440 (1948).

This paper considers the solution of the integrodifferential equation

$$(*)\quad \mu\frac{\partial\psi(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial\psi(r,\mu)}{\partial\mu}+\psi(r,\mu)=\tfrac{1}{2}\int_{-1}^{1}\psi(r,\mu')d\mu'=f(r),$$

which satisfies the boundary condition $\psi(a,\mu)=0$ for $0\leqq\mu\leqq 1$, where a is some assigned positive constant. If $f(r)$ is supposed known, equation (*) is a first order differential equation for ψ, and two independent integrals of the subsidiary equations are $r(1-\mu^2)^{\frac{1}{2}}=c$ $(0<c<\infty)$ and

$$\psi(r,\mu)e^{\pm(r^2-c^2)^{\frac{1}{2}}}\mp\int^r yf(y)(y^2-c^2)^{-\frac{1}{2}}e^{\pm(r^2-c^2)^{\frac{1}{2}}}dy=h_\pm(c),$$

where the $\pm$ refer to the sign of μ. Using these integrals, the author writes down a formal solution of the problem. The solution obtained in this manner shows that the point $\mu=0$, $r=a$ $(c=a)$ is a singular point of ψ and that the function is essentially indeterminate at this point. The author relates this singularity to the "shadow" of the sphere $r=a$.

S. Chandrasekhar (Williams Bay, Wis.).

Davison, B. Influence of a small black sphere upon the neutron density in an infinite non-capturing medium. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-88** (N.R.C. 1549), i+40 pp. (1944).

It can be shown that, for the problem considered in the preceding review, the function $f(r)=\frac{1}{2}\int_{-1}^{1}\psi(r,\mu)d\mu$ satisfies the linear integral equation

$$(*)\quad rf(r)=\tfrac{1}{2}\int_a^\infty tf(t)\{E_1(|r-t|)-E_1[(t^2-a^2)^{\frac{1}{2}}+(r^2-a^2)^{\frac{1}{2}}]\}dt,$$

where E_n denotes the exponential integral of order n. Writing the solution for $f(r)$ in the form $f(r)=r^{-1}f(\infty)[r+q(r)]$, the author derives the integral equation

$$q(r)=\tfrac{1}{2}\int_a^\infty q(t)\{E_1(|r-t|)-E_1[(r^2-a^2)^{\frac{1}{2}}+(t^2-a^2)^{\frac{1}{2}}]\}dt+\bar{E}_3(r,a),$$

where $\bar{E}_3(r,a)=\frac{1}{2}\{E_3(r-a)-E_3[(r^2-a^2)^{\frac{1}{2}}]-aE_2(r-a)\}$ $(r\geqq a)$. For small values of a, a perturbation method is developed and the following expansion for $q(\infty)$ is derived:

$$q(\infty)=-\tfrac{3}{4}a^2+\tfrac{1}{4}a^3-\tfrac{3}{8}(\tfrac{1}{4}\pi^2-1)a^4\log a+Ka^4+O(a^5),$$

where K is a certain numerical constant which is evaluated.

S. Chandrasekhar (Williams Bay, Wis.).

Davison, B. Influence of a large black sphere upon the neutron density in an infinite non-capturing medium. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-93** (N.R.C. 1550), i+15 pp. (1944).

In this paper the solution of equation (*) of the preceding review is considered for large values of a. In this case it is convenient to write the solution in the form $rf(r)=q(x)f(\infty)$, where $r=a+x$, and consider the equation

$$q(x)=\tfrac{1}{2}\int_0^\infty q(y)E_1(|x-y|)dy - \tfrac{1}{2}\int_0^\infty q(y)E_1[(x^2+2ax)^{\frac{1}{2}}+(y^2+2ay)^{\frac{1}{2}}]dy.$$

For large values of a, the second term on the right hand side can be treated as a correction and the solution obtained by an iteration scheme starting with the first approximation $q_0(x)$ which is the solution of the equation

$$q_0(x)=\tfrac{1}{2}\int_0^\infty q_0(y)E_1(|x-y|)dy.$$

In this manner, the author finds that

$$\lim_{x\to\infty}[q(x)-x]=\lim_{x\to\infty}[q_0(x)-x]-\tfrac{1}{4}a^{-2} -\tfrac{1}{4}a^{-3}\log a-\tfrac{1}{2}a^{-3}(c_1+\gamma-\tfrac{1}{4})+O(a^{-4}\log^2 a),$$

where $c_1=1.278794$ and γ is Euler's constant. It is known that the first term on the right hand side is $0.7104\cdots$.

S. Chandrasekhar (Williams Bay, Wis.).

Collatz, L. Eigenwertaufgaben bei einer Klasse linearer Integro-Differentialgleichungen. Z. Angew. Math. Mech. 25/27, 129–130 (1947).

The paper is a résumé without proofs of the author's results concerning eigenvalue problems of the form (1) $\phi(y)=\lambda\psi(y)$, where

$$\phi(y)\equiv M(y)+k\int_a^b F(x,\xi)y(\xi)d\xi,$$

$$\psi(y)\equiv N(y)+k\int_a^b G(x,\xi)y(\xi)d\xi;$$

$M(y)$ and $N(y)$ are linear differential expressions of order m and n, respectively ($m>n$), and, in addition to (1), the unknown function y is subject to m linearly independent boundary conditions. *E. H. Rothe* (Ann Arbor, Mich.).

Functional Analysis

Nakano, Hidegorô. On the product of relative spectra. Ann. of Math. (2) 49, 281–315 (1948).

The author applies his theory of relative spectra [Proc. Phys.-Math. Soc. Japan (3) 23, 485–511 (1941); these Rev. 3, 210] to prove and extend a theorem of M. H. Stone concerning rings of real-valued functions continuous on a compact space S. According to Stone's announcement [Proc. Nat. Acad. Sci. U. S. A. 26, 280–283 (1940); these Rev. 1, 338] such a ring which includes all constant functions $f(x)=r$ (r rational) can be characterized as an abstract partially ordered ring $\mathfrak{R}$ with the properties: (1) $\mathfrak{R}$ has a unit e; (2) $\mathfrak{R}$ is Archimedean: $na\leqq e$ for all positive integers n implies $a\leqq 0$; (3) its elements are bounded: for every a there is an n such that $a\leqq ne$; (4) $na=e$ has a solution a for every positive integer n; (5) $a^2\geqq 0$ for all a; (6) $\mathfrak{R}$ is commutative: $ab=ba$ for all a, b. The author shows that (6) is a consequence of the other properties. By assuming that $\mathfrak{R}$ admits the real numbers as multipliers, and that $\mathfrak{R}$ is a directed set, i.e., any a, b imply the existence of an element c such that $a\leqq c$ and $b\leqq c$, (4) and (5) can be omitted.

Now let S be any locally-compact space and call a continuous function $f(x)$ a C-function if, for every $\epsilon>0$, the set of x for which $|f(x)|\geqq\epsilon$ is a compact subset of S. Then the rings of C-functions which include $cf(x)$ with $f(x)$ for all real numbers c can be characterized as abstract partially ordered rings which admit the real numbers as multipliers, which are directed sets, and satisfy (1′) $na\leqq p$ for all positive integers n, and $p\geqq 0$, imply $a\leqq 0$; (2′) for every $a\geqq 0$ there is a positive number α such that $ax\leqq\alpha x$ and $xa\leqq\alpha x$ for all $x\geqq 0$; (3′) if both $ax\geqq 0$ and $xa\geqq 0$ hold for every $x\geqq 0$ then $a\geqq 0$. The property (3′) can be replaced by (3″) for every $a\geqq 0$ there are b, c such that $b\geqq 0$, $c\geqq 0$ and $a\leqq bc$. As in Stone's theorem, the ring $\mathfrak{R}$ can be represented by a ring of C-functions on a locally compact S' such that every C-function on S' can be uniformly approximated by elements of $\mathfrak{R}$, and this property determines S' uniquely up to topological equivalences. *I. Halperin.*

Nakano, Hidegorô. Reduction of Bochner's theorem to Stone's theorem. Ann. of Math. (2) 49, 279–280 (1948).

The author reverses the usual procedure and shows that Bochner's integral representation of positive definite functions follows from Stone's spectral theorem for separable Hilbert space. *R. A. Leibler* (Washington, D. C.).

Šilov, G. On normed rings possessing one generator. Rec. Math. [Mat. Sbornik] N.S. 21(63), 25–47 (1947). (Russian. English summary)

A normed ring R possesses one generator x if it is equal to the norm closure of polynomials in x. Then R can be represented as a ring of continuous functions defined on a bounded closed set S in the complex plane. In fact, S is homeomorphic to the space $\mathfrak{M}$ of maximal ideals in R and coincides with the range of $x(M)$ where M ranges over $\mathfrak{M}$ [Gelfand, same Rec. (N.S.) 9(51), 3–24 (1941); these Rev. 3, 51]. The following results are obtained. (1) A set S in the plane is the space of maximal ideals of a ring with one generator if, and only if, it is bounded, closed and does not separate the plane. (2) All functions analytic on S are contained in R. However, if λ_0 is a nonisolated point of S, then there exists $w(z)\in R$ which is not analytic at λ_0. (3) The set of all functions analytic on an infinite bounded closed set S cannot constitute a ring with S as its space of maximal ideals. (4) If S is a bounded, closed set which does not separate the plane and if $\lambda_0\in S$ is such that there exists a circle $Q\subset S$ with λ_0 on its boundary, then for $w(z)$ not analytic at λ_0 there exists a ring with one generator and space of maximal ideals S which does not contain $w(z)$. (5) A necessary and sufficient condition that a ring with one generator x be contained topologically as a subring of a ring in which x is regular is that x not be a generalized divisor of zero [Šilov, C. R. (Doklady) Acad. Sci. URSS (N.S.) 29, 83–84 (1940); these Rev. 2, 314]. [The problem of ring extensions which regularize elements which are not generalized divisors of zero has also been considered by the reviewer, Duke Math. J. 14, 1063–1077 (1947); these Rev. 9, 358.] (6) For a special class of rings $K_{<a_n>}$, S is a circle $|\lambda|\leqq r$ and there exists $0\leqq r_0\leqq r$ such that $z-\lambda e$ is or is not a generalized divisor of zero according as $r_0\leqq|\lambda|\leqq r$ or

$|\lambda| < r_b$. (Here $\alpha_0 = 1$, $\alpha_{m+n} \leqq \alpha_m \alpha_n$ and $K_{<\alpha_n>}$ consists of all $y = \sum_0^\infty a_n z^n$ for which $\|y\| = \sum |a_n| \alpha_n < \infty$.)

C. E. Rickart (New Haven, Conn.).

Rohlin, V. Unitary rings. Doklady Akad. Nauk SSSR (N.S.) 59, 643–646 (1948). (Russian)

A unitary ring $\mathfrak{H}$ is a complex Hilbert space in which, for certain pairs of elements, an operation of multiplication is defined satisfying: (1) if fg is defined then $gf = fg$; (2) if fg, $(fg)h$ and gh are defined, then $f(gh) = (fg)h$; (3) if fh and gh are defined and α, β are complex numbers, then $(\alpha f + \beta g)h = \alpha(fh) + \beta(gh)$; (4) there exists $e \in \mathfrak{H}$ such that $ef = f$ for all $f \in \mathfrak{H}$; (5) if $f_n g$ are defined and $f_n \to f$, $f_n g \to h$, then $fg = h$; (6) the set $\mathfrak{M}$ of all $f \in \mathfrak{H}$ such that fg is defined for all $g \in \mathfrak{H}$ is dense in $\mathfrak{H}$; moreover if fg is defined then there exist $f_n \in \mathfrak{M}$ such that $f_n \to f$ and $f_n g \to fg$; (7) for every $f \in \mathfrak{H}$ there exists $\bar{f} \in \mathfrak{H}$ such that $(fg, h) = (g, \bar{f}h)$ for all $f, g \in \mathfrak{M}$. Subrings and ideals of $\mathfrak{H}$ are defined in the obvious way. A one-to-one mapping U of $\mathfrak{H}$ into $\mathfrak{H}'$ is called an "isomorphism" provided it is linear, $(Uf, Ug) = (f, g)$ and $U(fg) = Uf \cdot Ug$, where $Uf \cdot Ug$ is defined if and only if fg is defined. If $\mathfrak{H} = \mathfrak{H}'$, then U is an "automorphism." Two automorphisms, U of $\mathfrak{H}$ and U' of $\mathfrak{H}'$, are "isomorphic" provided there exists an isomorphism V of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $U' = VUV^{-1}$. A subring $\mathfrak{R}$ of $\mathfrak{H}$ is "invariant" with respect to an automorphism U of $\mathfrak{H}$ provided $U(\mathfrak{R}) \subseteq \mathfrak{R}$.

Let M denote a Lebesgue space [see two previous notes by the author, same Doklady (N.S.) **58**, 29–32, 189–191 (1947); these Rev. **9**, 230, for the definition of Lebesgue space as used here and other notions referred to below] and denote by $\mathfrak{H}(M)$ the Hilbert space of complex functions (modulo sets of measure zero) whose absolute values are square integrable on M. If multiplication is defined as ordinary multiplication of functions, when it gives an element of $\mathfrak{H}(M)$, and if $\bar{f}$ denotes the complex conjugate of f, then $\mathfrak{H}(M)$ becomes a unitary ring. Conversely, every unitary ring is isomorphic to an $\mathfrak{H}(M)$ for some M. The theory of unitary rings is equivalent to the theory of Lebesgue spaces M. An ideal in $\mathfrak{H}(M)$ consists of all functions which vanish outside a certain measurable subset of M and conversely. A subring $\mathfrak{R}$ of $\mathfrak{H}(M)$ consists of all functions constant on elements of a certain "measurable decomposition" ζ of M, and conversely. An "isomorphism" of a Lebesgue space M into a second such space M' induces a "conjugate" isomorphism of $\mathfrak{H}(M')$ into $\mathfrak{H}(M)$ and conversely. Automorphisms U, U' of $\mathfrak{H}(M)$, $\mathfrak{H}(M')$, respectively, are isomorphic if and only if their associated "automorphisms" T, T' of M, M', respectively, are of the same "type." A decomposition ζ of M is said to be "invariant" with respect to T if the image of each element of ζ is contained in an element of ζ. The decomposition ζ is invariant with respect to an automorphism T of M if and only if the associated subring $\mathfrak{R}(\zeta)$ is invariant with respect to the conjugate automorphism U of $\mathfrak{H}(M)$ determined by T. Two examples are discussed. No proofs are given. *C. E. Rickart.*

Cooper, J. L. B. Symmetric operators in Hilbert space. Proc. London Math. Soc. (2) **50**, 11–55 (1948).

The objective of the present paper is to study symmetric operators H by solving the differential equation $dU/dt = iHU$. In differentiating, the limiting process may be either strong, dU/dt, or weak, $\delta U/\delta t$. It is shown that the weak element equation $\delta\psi/\delta t = iH\psi(t)$, $\psi(0) = \varphi$, has a solution for every $\varphi \in \mathfrak{D}$ (the domain of H), and if H is self-adjoint, we may substitute the strong derivative $d\psi/dt$ in place of $\delta\psi/\delta t$. In the self-adjoint case, $U(t)\varphi = \psi(t)$ defines the usual Stone group $U(t) = \exp(iHt)$ and this of course yields the spectral theory. Maximal symmetric operators are divided into two classes, accordingly as unique solutions of the above weak element equation hold for all $t \geqq 0$ or all $t \leqq 0$ and this is correlated with the usual deficiency index discussion. In the first class, a $U(t)$ is obtained for all $t \geqq 0$ such that $U(t_1)U(t_2) = U(t_1 + t_2)$ for all $t_1 \geqq 0$, $t_2 \geqq 0$. For $t \leqq 0$, a $V(t)$ is obtained such that $U^*(t) = V(-t)$. Then $U(t)$ commutes with H, V with H^*. The transformation $U(t)$ is isometric with domain $\mathfrak{H}$. The projection $E(t)$ on the range of $U(t)$ is introduced. The range of $E(\infty)$ is such that it contains that part of H which has a self-adjoint character. With this split off, the remainder of H is expressed as a "sum" of irreducible operators, each congruent to the operator of index one, and the structure of this atomic operator expressed in terms of $U(t)$. Thus the theory presented yields all the essential results obtained originally by means of the Cayley transform and the author points out that here the physical interpretation of the quantum mechanical requirement that the H in Schrödinger's equation be self-adjoint is immediately seen to be equivalent to the statement that the system has a complete past and a complete future.

F. J. Murray (New York, N. Y.).

Livšic, M. S. On the theory of isometric operators with equal deficiency indices. Doklady Akad. Nauk SSSR (N.S.) **58**, 13–15 (1947). (Russian)

Let H be a Hilbert space and let V be an isometric operator with domain G and range G' such that, if $D = H - G$ and $D' = H - G'$, then the dimensions of D and D' are both equal to $m (\leqq \infty)$. If, for each complex number z, with $|z| < 1$, $V(z) = (V - z)(1 - \bar{z}V)^{-1}$, then $V(z)$ satisfies the same conditions as V. Let $\{g_j\}$ and $\{g_j'\}$, $j = 1, \cdots, m$, be orthonormal bases in D and D' and let $\{g_i(z)\}$, $i = 1, \cdots, m$, be a (not necessarily orthonormal) basis in $D(z)$. The characteristic function $w(z)$ of V is by definition the matrix function

$$w(z) = [(g_i(z), g_j)]^{-1}[(g_i(z), g_j')].$$

The author gives a necessary and sufficient condition that a prescribed function w be the characteristic function of some V, and states that two simple isometric operators V_1 and V_2 (i.e., operators which are not unitary on any subspace of H) are unitarily equivalent if and only if there exist two unitary matrices u' and u'' such that $w_1(z) = u' w_2(z) u''$.

If T is an extension of V such that $T(D) \subset D'$, then T is called a quasi-unitary operator of order m. Such an operator T is completely characterized by V and the matrix $u = [u_{ij}]$, where $Tg_i = \sum_{j=1}^m u_{ij} g_j'$. The characteristic function of T is by definition

$$w_T(z) = (1 - w(z)u^*)(u - w(z)).$$

If $m = 1$ and if $w_T(z) = w_1(z)w_2(z)$, where w_1 and w_2 are both conformal maps of the unit circle into itself, then w_1 is called a divisor of w_T. The author's last result asserts the existence of a natural one-to-one correspondence between the divisors of w and the invariant subspaces of T. The paper contains no proofs. *P. R. Halmos* (Chicago, Ill.).

Livšic, M. S. On the theory of the elementary divisors of non-Hermitian operators. Doklady Akad. Nauk SSSR (N.S.) **60**, 17–20 (1948). (Russian)

An operator A on a Hilbert space H is quasi-Hermitian if $A - A^*$ is finite dimensional. The author remarks that by means of the Cayley transform the study of quasi-Hermitian operators reduces to the study of quasi-unitary operators which he has discussed in a previous paper [cf. the preceding review]. A quasi-unitary operator T is a contraction if

$\|Tf\| \leqq \|f\|$ for all f in H. (I) If T is a contraction, then $H = H_0 \oplus (\overline{H_1 + H_2})$, where H_i is invariant under T, $i = 0, 1, 2$; T on H_0 is unitary; H_1 is the closed linear manifold spanned by all proper vectors of T; and H_2 has no finite dimensional invariant subspaces. (II) If B is completely continuous and the imaginary part of B is finite dimensional and non-negative, then $H = \overline{H_1 + H_2}$, where H_i is invariant under T, $i = 1, 2$; H_1 is the closed linear manifold spanned by all proper vectors of B; H_2 has no finite dimensional invariant subspaces; and there exists in H_2 a continuous family of linear manifolds, strictly increasing from (0) to H_2. An application to differential operators is described. The paper contains no proofs. *P. R. Halmos* (Chicago, Ill.).

Krasnosel'skiĭ, M. A. On the extension of Hermitian operators with a nondense domain of definition. Doklady Akad. Nauk SSSR (N.S.) **59**, 13–16 (1948). (Russian)

A Hermitian operator A on a unitary space $\mathfrak{H}$ is called semimaximal if it has no Hermitian extension $\tilde{A}$ for which $\mathfrak{D}(\tilde{A}) \subset \overline{\mathfrak{D}(A)}$. Let $\mathfrak{B}$ be the orthogonal complement of $\overline{\mathfrak{D}(A)}$, and let $P(\lambda)$ be the projection operator whose range is the defect subspace $\mathfrak{N}(\lambda)$. Write $\mathfrak{M}(\lambda) = P(\lambda)\mathfrak{B}$ and define an operator V on $\mathfrak{M}(\lambda)$ by $VP(\lambda)f = P(\bar{\lambda})f$, $f \in \mathfrak{B}$. A semidefect index of A is by definition the dimension of the orthogonal complement in $\mathfrak{N}(\lambda)$ of $\overline{\mathfrak{M}(\lambda)}$. In terms of these notations and definitions the author states (without proof) the following results. (1) If A is closed and has equal defect indices, and if $\tilde{A}$ is a self-adjoint extension of A, then $\tilde{A}(g + f - Uf) = Ag + \bar{\lambda}f - \lambda Uf$, $g \in \mathfrak{D}(A)$, $f \in \mathfrak{N}(\lambda)$, where U is an isometric mapping of $\mathfrak{N}(\lambda)$ on $\mathfrak{N}(\bar{\lambda})$ such that $Uf \neq Vf$ for every f in $\mathfrak{M}(\lambda)$. (2) If A is closed, then a necessary and sufficient condition that A have a semimaximal extension is that V be closed. (3) If A and V are both closed, then a necessary and sufficient condition that A be semimaximal is that it have equal semidefect indices. (4) If A is closed and $\mathfrak{D}$ is a subspace of $\mathfrak{D}(A)$, then A has a closed Hermitian extension $\tilde{A}$ for which $\overline{\mathfrak{D}(\tilde{A})} = \mathfrak{D}$.

P. R. Halmos (Chicago, Ill.).

Neumark, M. A. Extremal spectral functions of a symmetric operator. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 327–344 (1947). (Russian. English summary)

Detailed proofs of some results the author announced before [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 7–9 (1946); these Rev. **8**, 386]. *P. R. Halmos* (Chicago, Ill.).

Citlanadze, È. S. On a question about characteristic values of nonlinear completely continuous operators in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **57**, 879–881 (1947). (Russian)

This paper is a sequel to an earlier one by the same author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 15–18 (1947); these Rev. **9**, 95, where the author's name was transliterated Zitlanadze], and is concerned with the existence and topological structure of the set of eigen-elements of certain nonlinear operators in Hilbert space. Let S be the unit sphere $\|x\| \leqq 1$ in a real Hilbert space, S_1 its "surface" $\|x\| = 1$, and S_1^* the projective sphere derived from S_1 by identifying antipodal points. Let f be a weakly continuous functional defined on S, such that $f(a) = f(-a)$ for all a, and suppose that f possesses a Fréchet differential throughout S, generating the (not necessarily linear) operator L. Suppose also that L is bounded and satisfies a Lipschitz condition on S. Write $K(a) = La - (La, a)a$. Then an element a of S_1 is an eigen-element of L if $\|K(a)\| = 0$, and it is said to be an α-eigen-element of L if $\|K(a)\| \leqq \alpha$. Now let $[A]$ be a "topological class" of sets in S_1^*, i.e., a class of closed sets with the property that with a set A it also contains all sets into which A can be deformed. Write $C(A) = \inf f(x)$ for $x \in A$ and $c = \sup C(A)$ for $A \in [A]$. Then, for any $\epsilon > 0$, there is a set $A_\epsilon \in [A]$ such that $C(A_\epsilon) > c - \epsilon$.

The author begins by proving that for any $\alpha > 0$ there is an α-eigen-element a_α of L in A_ϵ, such that $c - \epsilon \leqq f(a_\alpha) \leqq c + \epsilon$. Under the additional assumption that for any $d > 0$ there is a number $\lambda_d > 0$ such that $f(a) \geqq d$ implies $\|La\| \geqq \lambda_d$, he then states the following results without proof. (1) There exist eigen-elements a of L such that $f(a) = c$. (2) The set T_0 of such eigen-elements is compact. (3) Let $[A_k]$ be the topological class of the bounded compact sets on S_1^* whose Lusternik-Schnirelmann [L.-S.] category is greater than or equal to k, and let c_k be the least upper bound associated with $[A_k]$ in the way described above. (A closed set has L.-S. category n if it can be expressed as the union of n and no fewer closed sets, each deformable into a point.) Then, if $c_i = c_j$ for two indices i, j such that $j > i$, the L.-S. category of T_0 will be greater than or equal to $j - i + 1$.

F. Smithies (Cambridge, England).

Aronszajn, Nachman. Les noyaux pseudo-reproduisants. C. R. Acad. Sci. Paris **226**, 456–458 (1948).

Aronszajn, Nachman. Noyaux pseudo-reproduisants et complétion des classes hilbertiennes. C. R. Acad. Sci. Paris **226**, 537–539 (1948).

Aronszajn, Nachman. Complétion fonctionnelle de certaines classes hilbertiennes. C. R. Acad. Sci. Paris **226**, 617–619 (1948).

Aronszajn, Nachman. Propriétés de certaines classes hilbertiennes complétées. C. R. Acad. Sci. Paris **226**, 700–702 (1948).

The author has introduced [Proc. Cambridge Philos. Soc. **39**, 133–153 (1943); these Rev. **5**, 38] the concept of reproducing kernel for a Hilbert space represented as a class of functions F defined and finite everywhere on a set E. The present notes concern a generalization of the concept to the case where E is a measure space and the functions $f(x)$ of F are defined modulo a σ-ideal $\mathfrak{A}$ of sets of measure 0. The measure μ and a fundamental system of intervals $U(x, \rho)$ (the open spheres with center x and radius ρ in the Euclidean case) are fixed, but $\mathfrak{A}$ may vary between the limits $\mathfrak{A}^0$ (the empty set) and $\mathfrak{A}_\mu$ (all sets of μ-measure 0). The complete definition of a pseudo-reproducing kernel of F relative to $\mathfrak{A}$ is too lengthy to reproduce. Essentially it is a function $N(x, y)$ on $E \times E$, satisfying certain summability conditions, such that for every $f \in F$

$$f(y) = \lim_{\rho \to 0} \frac{1}{\mu[U(y, \rho)]} \left[f(x), \int_{U(y,\rho)} N(x, z) d\mu_z \right]$$

for all y (with the exception of a set of $\mathfrak{A}$), where $[f, g]$ is the scalar product. If a pseudo-reproducing kernel exists it is essentially unique mod $\mathfrak{A}$, and its existence is equivalent to the absolute continuity of a certain set function on $E \times E$ induced by F. A pseudo-reproducing kernel is the kernel of a positive definite integral operator.

In general one starts with an incomplete space F with a formal pseudo-reproducing kernel N relative to $\mathfrak{A}$ ($\mathfrak{A} = \mathfrak{A}^0$ usually) and tries to find an ideal $\bar{\mathfrak{A}} \supset A$ and a complete space $\bar{F} \supset F$ of functions defined with the exception of a set of $\bar{\mathfrak{A}}$ such that (a) F is dense in $\bar{F}$; (b) $f_n \to f$ strongly in F implies the existence of a subsequence $f_{n_i}(x) \to f(x)$ for all x

with the exception of a set of $\mathfrak{N}$. Moreover, one wishes N to be a pseudo-reproducing kernel of F relative to $\mathfrak{N}$. The central problem is to find the smallest class $\mathfrak{N} \supset \mathfrak{A}$ for which this is possible.

Applications to potential theory and biharmonic functions are sketched. For example, let E be a domain in Euclidean n-space with sufficiently regular frontier C, and let F consist of the functions $f(x)=f(x_1, \cdots, x_n)$ of class C' in $E+C$ such that (a) $[f, f]=\int_E \sum |f_{x_k}|^2 dx < \infty$; (b) $f=0$ on C or $\int_C f d\sigma = 0$. The Green's or Neumann's function is then a pseudo-reproducing kernel for F relative to $\mathfrak{A}^0$. The ideal $\mathfrak{N}$ may be taken as at worst $\mathfrak{A}_{n-2}$ ($n \geqq 2$), where $\mathfrak{A}_\alpha$ denotes the class of sets of null $(\alpha+\epsilon)$-dimensional Hausdorff measure. The completion $\bar{F}$ relative to $\mathfrak{A}_{n-2}$ is characterized. Analogous results for biharmonic functions are stated. The ultimate applications make contact with results of Calkin and Morrey [Duke Math. J. 6, 170–186, 187–215 (1940); these Rev. 1, 208, 209] and Friedrichs [Duke Math. J. 14, 67–82 (1947); these Rev. 8, 583]. No proofs are given.

W. F. Eberlein (Princeton, N. J.).

Katětov, Miroslav. On normed vector spaces. Rozpravy II. Třídy České Akad. 53, no. 45, 27 pp. (1943). (Czech)

Katětov, Miroslav. On the theory of topological vector spaces. Rozpravy II. Třídy České Akad. 53, no. 46, 12 pp. (1943). (Czech)

German summaries appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 44, 594–598, 599–605 (1943); these Rev. 8, 468, 519.

Zaanen, A. C. On linear functional equations. Nieuw Arch. Wiskunde (2) 22, 269–282 (1948).

The author discusses a theorem of Schauder [Studia Math. 2, 183–196 (1930)] which states that the equations $x-T(x)=0$ and $x^*-T^*(x^*)=0$ have the same number of linearly independent solutions, where T is a completely continuous linear transformation on a Banach space to itself and T^* is its adjoint. In the course of this discussion he gives a proof of the theorem which does not employ the result of Schauder [loc. cit.] that if T is completely continuous so is T^*. *B. Yood* (Ithaca, N. Y.).

Èzrohi, I. A. The general form of linear operations in spaces with a countable basis. Doklady Akad. Nauk SSSR (N.S.) 59, 1537–1540 (1948). (Russian)

Let Z be a Banach space which admits a basis $\{z_i\}$ such that the expansion $z=\sum_{i=1}^{\infty} z_i \psi_i(z)$ (where each ψ_i is a linear functional) has the property that $\|\sum_{i=1}^{n} z_i \psi_i(z)\|$ is non-decreasing as a function of n for each $z \in Z$. Let X be an arbitrary Banach space. The author observes that the general linear transformation from X to Z has the form $u(x)=\sum_{i=1}^{\infty} z_i f_i(x)$, where each f_i is a linear functional on X and the series converges for all x in X and that

$$\|u\| = \lim_{n \to \infty} \sup_{\|x\| \leq 1} \|\sum_{i=1}^{n} z_i f_i(x)\|.$$

He makes a similar observation for the case in which the roles of X and Z are interchanged. *G. W. Mackey.*

Marinescu, G. Sur l'extension des fonctionnelles bilineaires. Bull. Math. Soc. Roumaine Sci. 47, 202–209 (1946).

[In the original, "extension" appears as "extensinon."] The author observes that a bilinear functional F defined on a linear manifold in a Banach space E is a functional defined on a linear manifold in $E \times E$, the direct product in the sense of Schatten [Trans. Amer. Math. Soc. 53, 195–217 (1943); these Rev. 4, 161]. Then if F satisfies an appropriate condition involving a crossnorm (which makes $E \times E$ a Banach space) the theory of the extension of linear functionals may be applied to F. *B. Yood* (Ithaca, N. Y.).

Brodskiĭ, M. S., and Mil'man, D. P. On the center of a convex set. Doklady Akad. Nauk SSSR (N.S.) 59, 837–840 (1948). (Russian)

Let K be a convex bounded set in a Banach space B. A point p_0 in K is diametral if $\sup_{p \in K} \|p_0 - p\| = d$, the diameter of K. The set K is said to have normal structure if every convex subset of K which contains more than one point contains at least one nondiametral point. A sequence $\{x_n\}$ of points of B is called a diametral sequence if the distance d_n from x_{n+1} to the convex hull of the x_i, $i \leqq n$, tends to the diameter d of the set of all x_i. (The unit vectors in l_1 form a diametral sequence.) Theorem 1. K has normal structure if and only if it contains no diametral sequence. Let $r(B)$ be the maximum of those numbers r for which for every integer n there exists an n-dimensional simplex in B with all edge lengths arbitrarily near r and with all its points, both interior and boundary, arbitrarily near norm one. Theorem 2. If $r(B)<1$, then every convex set in B has normal structure.

Assume now that K has normal structure and is compact in some topology in which spheres are closed. [This note gives explicitly the case where B is a conjugate space and K is transfinitely closed.] Then the following construction gives a uniquely determined point, called the center of K. Let K_ϵ be the intersection of all closed spheres of radius $d-\epsilon$ centered at points of K; normal structure shows that there exists $\epsilon > 0$ such that K_ϵ is nonempty. Compactness shows that, if $\epsilon' = \sup \epsilon$ with K_ϵ nonempty, then $K_{\epsilon'}$ is also nonempty; call it K^1. Transfinite induction using this step for nonlimit ordinals and intersections for limit ordinals gives a transfinite sequence $K^1 \supseteq \cdots \supseteq K^\theta \supseteq \cdots$; normal structure asserts that these inclusions are proper until a θ is reached with K^θ containing only one point, the center of K. This name is justified by theorem 3: the center of a convex bounded K for which the above construction can be carried through is a fixed point of every isometric transformation of K into itself. Theorem 4. If K is also compact in the norm topology, the center is a fixed point of every continuous transformation of K into itself which does not decrease distances. *M. M. Day* (Princeton, N. J.).

Mil'man, D. P., and Rutman, M. A. On a more precise theorem about the completeness of the system of extremal points of a regularly convex set. Doklady Akad. Nauk SSSR (N.S.) 60, 25–27 (1948). (Russian)

The authors sharpen the known theorem [Krein and Šmulian, Ann. of Math. (2) 41, 556–583 (1940); these Rev. 1, 335] that every bounded, regularly convex (i.e., w^*-closed and convex) subset K of the conjugate space E^* of a Banach space E is the w^*-closed convex hull of its set of extreme points. For such a K and for g in K define $M(g)$ to be the set of all x of norm one such that $g(x)=\sup f(x)$ for $f \in K$. The w^*-compactness of K shows that every x of norm one is in an $M(g)$. Call $M(g)$ maximal if it is not a proper subset of an $M(g_1)$, g_1 in K. Lemma 1. Every $M(g)$ is contained in a maximal $M(g_1)$. If M is a maximal $M(g)$, define the extremal face $K(M)$ of K to be the set of all f in K for which

$M(f)=M$. Lemma 2. Every extremal face $K(M)$ of K is regularly convex and each extreme point of each $K(M)$ is an extreme point of K. Theorem. Let K be regularly convex and bounded and let A consist of just one point from each extremal face of K. Then the regularly convex hull of A coincides with K; moreover, each element of A can be chosen as an extreme point of K. *M. M. Day* (Princeton, N. J.).

Mil'man, D. Accessible points of a functional compact set. Doklady Akad. Nauk SSSR (N.S.) **59**, 1045–1048 (1948). (Russian)

Let Q be a set, let m_Q be the Banach space of all bounded, real-valued functions on m_Q and let R be a linear subspace of m_Q containing all constant functions and enough other functions so that, if $q_1 \neq q_2$, there exists x in R with $x(q_1) \neq x(q_2)$. Imbed Q in R^*, the conjugate space of R, by the natural mapping $q \leftrightarrow f_q$ if and only if $f_q(x)=x(q)$ for every x in R. The w^* topology in R^* imposes a topology on the set Q; call the resulting topological space Q_R. Let K_R be the w^*-closed convex hull of Q in R^*. In a preceding note [same Doklady **57**, 119–122 (1947); these Rev. **9**, 192] Mil'man defined the T-frontier of Q_R and proved that, if Q_R is bicompact, then the T-frontier of Q_R is the closure in Q_R of the set of extreme points of K_R. The point q_0 is an accessible point of Q_R if there exists x in R such that $x(q_0) > x(q)$ if $q \neq q_0$. Theorem 1. If Q_R is compact, metrizable, that is, if Q_R is bicompact and R is separable, then the set of accessible points of Q_R is dense in the T-frontier of Q_R. (Without metrizability Q_R need have no accessible points at all.) Theorem 2. If C_R is the subspace of m_Q containing just those x continuous on Q_R and if q_0 is an accessible point of Q_R, there exists a unique f in C_R^* such that $\|f\|=1$, $f(e)=1$ (where e is the constantly 1 function of C_R), and $f(x)=x(q_0)$ for all x in R.

These theorems are applied to the set Q of maximal ideals in a commutative Banach algebra R with a countable set of generators. An example is theorem 4. If M is an accessible maximal ideal and E is a closed set of maximal ideals not containing M (all regarded as imbedded in R^*), then the distance from M to E is 1. *M. M. Day* (Princeton, N. J.).

Mil'man, D. Dynamical systems defined by functionals and invariant measures on them. Doklady Akad. Nauk SSSR (N.S.) **59**, 1397–1398 (1948). (Russian)

Let Q be an abstract set, let E be a linear set of bounded real-valued functions on Q which contains the constant functions and let S be a semigroup of transformations of Q into Q which contains the identity transformation. Let Q be topologized as follows: a typical neighborhood of $q \in Q$ is the set of $q' \in Q$ satisfying $|x_i(q)-x_i(q')|<\epsilon$ $(i=1, \cdots, n; x_i \in E, \epsilon>0)$. Let E be normed as follows: $\|x\|=\sup_{q \in Q}|x(q)|$ $(x \in E)$. Assume: (1) $q, q' \in Qq \neq q'$ implies $x(q) \neq x(q')$ for some $x \in E$; (2) $s, s' \in S$ implies $s''Q \subset sQ \cap s'Q$ for some $s'' \in S$; (3) $x \in E$ implies that E contains the orbit of x (the set of translates of x by elements of S); (4) E is separable; (5) the orbit of each $x \in E$ is compact. A measure is understood to be a regular Borel measure in Q. A measure m is called invariant in case $m(s^{-1}I)=m(I)$ $(s \in S, I$ a Borel set) and normalized in case $m(Q)=1$. Let K be the set of normalized invariant measures; K is convex. Make the following definitions: $m, m' \in K$ are E-distinguishable in case $\int_Q x dm \neq \int_Q x dm'$ for some $x \in E$; $m \in K$ is E-accessible in case there exists $x \in E$ such that $\int_Q x dm' < \int_Q x dm$ for all $m' \in K$ which are E-distinguishable from m; $m \in K$ is extremal in case m is not the midpoint of any segment lying in K; $m \in K$ is a weak limit of a sequence $\{m_i; m_i \in K\}$ in case $\int_Q x dm_i \to \int_Q x dm$ for all continuous functionals x on Q. The following theorems are announced without proof: (1) K is nonvacuous; (2) K is the weak closure of the convex hull of M, where M is the set of E-accessible extremal points of K. *W. H. Gottschalk.*

Minkevič, M. I. The theory of integral funnels in generalized dynamical systems without a hypothesis of uniqueness. Doklady Akad. Nauk SSSR (N.S.) **59**, 1049–1052 (1948). (Russian)

This paper studies one-parameter systems of set-valued transformations with respect to properties generalizing those studied by G. D. Birkhoff [cf. his Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927, chap. 7] for one-parameter systems of point-valued transformations. Let R be a compact metric space with metric ρ. Let C be the collection of nonvacuous closed subsets of R and let C be metrized by the Hausdorff metric, say σ. Let $S: R \times T \to C$, where T denotes the set of real numbers. Assume that these conditions are satisfied: (1) $p \in R$ and $t, s \in T$ with $ts \geqq 0$ implies $S(S(p, t), s)=S(p, t+s)$; (2) if $p, q \in R$ and if $t \in T$, then $q \in S(p, t)$ implies $p \in S(q, -t)$; (3) the function $S(p, t)$ $(p \in R, t \in T)$ is continuous in t for each fixed p. A transformation S satisfying the above conditions is defined by a system of differential equations $x'=f(x)$ (x and $f(x)$ are n-dimensional vectors), where $f(x)$ is continuous.

If $p \in R$, then $S(p, T)$ is called the funnel of p, $S(p, T^+)$ is called the ω-funnel of p, where T^+ denotes the set of nonnegative real numbers, and $S(p, T^-)$ is called the α-funnel of p, where T^- denotes the set of nonpositive real numbers. A point p of R is called locally continuous provided that for some $\tau>0$ this property holds: if $|t| \leqq \tau$ and if $\epsilon>0$, then there exists $\delta>0$ such that $q \in R$ with $\rho(p, q)<\delta$ implies $\sigma(S(p, t), S(q, t))<\epsilon$. A point p of R is called integrally continuous provided that, if $\tau>0$ and if $\epsilon>0$, then there exists $\delta>0$ such that $q \in R$ with $\rho(p, q)<\delta$ and $|t| \leqq \tau$ implies $\sigma(S(p, t), S(q, t))<\epsilon$. If $p, q \in R$, then q is called an ω-limit point of p provided that, if $\epsilon>0$ and if $\tau>0$, then there exists $t>\tau$ such that $\rho(q, r)<\epsilon$ for some $r \in S(p, t)$. A set A in R is called ω-invariant provided that $p \in A$ implies $S(p, T^+) \subset A$. A set A in R is called ω-minimal provided that A is nonvacuous, closed, ω-invariant and does not contain a proper subset with these properties. A point p of R is called ω-recurrent provided that, if $\epsilon>0$, then there exists $L>0$ such that $t'>0$ and $t''>0$ implies the existence of τ $(0 \leqq \tau \leqq L)$ for which $\sigma(S(p, t'), S(p, t''+\tau))<\epsilon$. An α-limit point of a point, an α-invariant set, an α-minimal set and an α-recurrent point are defined dually.

The following theorems, together with the duals of 2, 3, 4, and 6, are announced without proof. (1) If $p \in R$ and if p and each point of $S(p, T)$ are locally continuous, then p and each point of $S(p, T)$ are integrally continuous. (2) If $p, q \in R$, if q is an ω-limit point of p and if q is integrally continuous, then each point of $S(q, T)$ is an ω-limit point of p. (3) If A is an ω-invariant subset of R and if each point of $\bar{A}-A$ is integrally continuous, then $\bar{A}$ is ω-invariant. (4) If A is a nonvacuous closed ω-invariant subset of R, then A contains an ω-minimal set. (5) If A is a nonvacuous closed subset of R such that the set of limit points of points in A is invariant, then A is a minimal set if and only if $p \in A$ implies that the closure of $S(p, T)$ is A. (6) If $p \in R$, if p is ω-recurrent and if the ω-limit points of p are integrally continuous, then the closure of $S(p, T^+)$ is an ω-minimal set.

W. H. Gottschalk (Philadelphia, Pa.).

Maier, A. G. On the ordinal number of central trajectories. Doklady Akad. Nauk SSSR (N.S.) **59**, 1393–1396 (1948). (Russian)

The author gives a constructive proof of the following statement. If α is an ordinal number of the second class, then there exists a dynamical system of class C^∞ in an open three-dimensional manifold such that $\alpha \leqq \beta$, where β is the ordinal number of the central trajectories.

W. H. Gottschalk (Philadelphia, Pa.).

Theory of Probability

Ballarin, Silvio. Espressione rigorosa dello scarto mediano nel problema delle prove ripetute nello schema di Bernoulli. Mem. Soc. Astr. Ital. (N.S.) **19**, 63–65 (1948).

Let S_n be the number of successes in n Bernoulli trials. An elementary computation shows that

$$E(|S_n - np|) = 2Kb(K, n)q,$$

where $K = [np] + 1$ and $b(K, n)$ is the probability of exactly K successes. *W. Feller* (Ithaca, N. Y.).

James, Hubert M. Statistical properties of networks of flexible chains. J. Chem. Phys. **15**, 651–668 (1947).

The main part of the paper treats properties of the multidimensional normal distribution which are well known to statisticians: it is shown that the marginal distributions are again normal, their parameters are determined, etc. These considerations are independent of the application to molecular chains. The second part is devoted to mechanical analogies which permit a calculation of the parameters of the author's chain model. *W. Feller* (Ithaca, N. Y.).

Gnedenko, B. V. On a theorem of S. N. Bernšteĭn. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 97–100 (1948). (Russian)

If a probability distribution in the plane has the property that in two different coordinate systems the coordinates are independent, then the distribution is normal. This generalizes a theorem attributed to S. Bernstein [Trudy Leningrad Politehn. Inst. **3**, 21–22 (1941)], to the effect that if X is independent of Y and $X+Y$ independent of $X-Y$, then X and Y are normally distributed. The proof is based on the use of characteristic functions.

W. Feller (Ithaca, N. Y.).

Tietze, Heinrich. Würfelspiel und Integralgeometrie. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 131–158 (1947).

Suppose that, instead of familiar dice, one uses an arbitrary convex polyhedron P. When thrown, P will come to rest on one of its n faces and the corresponding probabilities may in a natural way be defined as a function of the direction of P at the moment when P hits the plane. Assuming all directions equally probable the evaluation of the n probabilities reduces to the computation of certain contents and thus becomes a problem of integral geometry. This problem is treated in 2 and 3 dimensions. Numerous examples are given. *W. Feller* (Ithaca, N. Y.).

Richards, Paul I. Probability of coincidence for two periodically recurring events. Ann. Math. Statistics **19**, 16–29 (1948).

Consider two sequences of periodic events, each with fixed period and duration. The phases are unknown and treated as uniformly distributed random variables. It is desired to find the probability $p(t, \tau)$ that within time t the two events occur simultaneously and that their "overlap" is of at least duration τ. This probability is calculated, but it is extremely sensitive to small changes in periods and durations. From an experimental point of view it is therefore desirable to treat also periods and durations as random variables. The problem then becomes complicated and no attempt at a complete solution is made. However, usable practical approximations are obtained by a smoothing process which is difficult to describe in a few words. Graphs illustrate the practical results and permit one to estimate $p(t, \tau)$.

W. Feller (Ithaca, N. Y.).

Daudin, J. Calcul des observations de grandes gerbes. J. Phys. Radium (8) **8**, 301–305 (1947).

The integral

$$N \int_0^\infty \delta^{1-\gamma} \prod_{k=1}^{p} (1 - e^{-\lambda_k \delta}) e^{-\alpha \delta} d\delta$$

is introduced as an approximation to the probability that a system of p counters will register a shower. [The Poisson terms $1 - e^{-\lambda_k \delta}$ are discernible while the factor $\delta^{1-\gamma}$ is based on empirical evidence.] The integral is evaluated and experimental consequences are derived. *W. Feller.*

Horton, H. Burke. A method for obtaining random numbers. Ann. Math. Statistics **19**, 81–85 (1948).

Consider n roulette wheels each divided into two arcs and arranged in series so that the kth wheel determines which arc on the $(k+1)$th wheel will be called 0 and which 1. The final outcome of an n-tuple trial is 0 or 1 and the author shows that the corresponding probabilities tend to $\frac{1}{2}$ as $n \to \infty$, irrespective of the possible bias of the individual wheels. An unbiased passage from the dyadic to the decimal system is described. *W. Feller* (Ithaca, N. Y.).

Brunk, H. D. The strong law of large numbers. Duke Math. J. **15**, 181–195 (1948).

Let $\{X_n\}$ be a sequence of independent random variables with vanishing expectations. Kolmogorov proved that for $\{X_n\}$ to obey the strong law of large numbers it is sufficient that $\sum n^{-2} E(X_n^2) < \infty$. The author proves that, more generally, $\sum n^{-2p} E(X_n^{2p}) < \infty$ is sufficient. This condition is generalized to the case where the weighting factor n is replaced by an arbitrary sequence $\{a_n\}$. *W. Feller.*

Robbins, Herbert. On the asymptotic distribution of the sum of a random number of random variables. Proc. Nat. Acad. Sci. U. S. A. **34**, 162–163 (1948).

Let $\{x_k\}$ be a sequence of mutually independent random variables with a common fixed distribution function, and $E(x_k) = a$, Var $(x_k) = C^2$. Let N be an integral-valued positive random variable whose distribution depends on a parameter λ. Put $E(N) = \alpha = \alpha(\lambda)$, Var $(N) = \gamma^2 = \gamma^2(\lambda)$. The author studies the limiting distribution of the random variable $Y = X_1 + \cdots + X_N$ as $\lambda \to \infty$. This distribution is in general not normal. Let N^* and Y^* be the normalized variables N and Y, and let $\theta(t)$ and $\phi(t)$ be the corresponding characteristic functions. Put $\sigma^2 = \alpha C^2 + \gamma^2 a^2$ and suppose that as $\lambda \to \infty$, $\sigma \to \infty$, $\gamma = o(\sigma^2)$, $\gamma a/\sigma \to \delta$. Then $\phi(t) \to \theta(\delta t) \exp\{-t^2(1-\delta^2)/2\}$. It follows that the limiting distribution is normal whenever N^* is normally distributed and also if $a^2\gamma^2 = o(\alpha)$. On the other hand, if $a = 0$, $\gamma/\alpha \to r$ and if N^* has a limiting distri-

bution $g(x)$ then

$$\phi(t)\to\int_0^\infty e^{-\beta x/r}dg((x-1)/r).$$

These results are stated without proofs. *W. Feller.*

Sarymsakov, T. A., and Sultanova, M. The law of the iterated logarithm for Markov chains. Doklady Akad. Nauk SSSR (N.S.) **59**, 1249–1252 (1948). (Russian)

Let $\{X_k\}$ be a Markov process where all X_k have the same finite range of values. Put $S_n=\sum_{k=1}^n\{X_k-E(X_k)\}$. Let $m_1, m_2, \cdots$ be the consecutive times where $X_{m_r}=X_1$, and put $Y_r=S_{m_r}-S_{m_{r-1}}$. The Y_r are mutually independent and the authors show that $\sum \Pr\{|Y_r|>\nu^{\frac{1}{2}}\}<\infty$. Therefore the sequence $\{Y_r\}$ is equivalent to a sequence which satisfies the Kolmogorov conditions for the law of the iterated logarithm. Now let $\nu(N)$ be defined by $m_{\nu(n)}\leqq N<m_{\nu(n+1)}$. Then $S_n=Y_1+\cdots+Y_{\nu(n)}+\rho_n$ and it is easily seen that the remainder ρ_n can be neglected in the limit. The authors seem to forget that $\nu(n)$ is a random variable and that for given $\nu(n)$ the Y_k are no longer mutually independent. Except for this the above argument would imply the law of the iterated logarithm for S_n. [For a more complete treatment cf. W. Doeblin, Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples, Thèse, Paris, 1938; Bull. Math. Soc. Roum. Sci. **39**, no. 1, 57–115; no. 2, 3–61 (1937).] *W. Feller* (Ithaca, N. Y.).

Bityuckov, V. I. A local limit theorem for sequences of events forming a compound chain of the second order. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 101–110 (1948). (Russian)

Let $y_1, y_2, \cdots$ be a sequence of chance variables which can take only the values 0 and 1. They are assumed to constitute a Markov chain of the second order, i.e., the probability that $y_k=0$ or 1 is a function only of the values assumed by $y_{(k-2)}$ and $y_{(k-1)}$ (except of course when $k=1, 2$). All second order conditional probabilities are assumed neither zero nor one. Let $m(n)=\sum_{i=1}^n y_i$, and p and D be constants defined in the text (D implicitly in equation (15)). It is proved that the probability that $m(n)=np+xDn^{\frac{1}{2}}$ is $\pi^{-\frac{1}{2}}e^{-x^2}\Delta x[1+O(n^{-\frac{1}{2}})]$, where Δx is a suitably defined function of x and n. *J. Wolfowitz* (New York, N. Y.).

Sevast'yanov, B. A. On the theory of branching random processes. Doklady Akad. Nauk SSSR (N.S.) **59**, 1407–1410 (1948). (Russian)

On considère des particules de types $T_1, \cdots, T_n$. On désigne par $P_k^{(\alpha)}(t)$, $t=1, 2, \cdots$, la probabilité qu'une particule de type T_k se transforme, après t générations, en $\alpha_1, \cdots, \alpha_n$ particules de types respectifs $T_1, \cdots, T_n$. Donc $F_k(t; x)=\sum_\alpha P_k^{(\alpha)}(t)x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ est la t-ème itérée $f_k^{(t)}(x)$ de $f_k(x)=F_k(1; x)$. La probabilité de dégénérescence du processus commencé avec une particule de type T_k est: (a) $P_k^{(0,\cdots,0)}(t)$ à la t-ème génération, (b) $P_k=\lim_{t\to\infty}P_k^{(0,\cdots,0)}(t)$ à une génération quelconque. Le processus sera dit dégénérescent si tous les $P_k=1$. Comme les P_k vérifient le système d'équations $x_k=f_k(x_1, \cdots, x_n)$, $k=1, \cdots, n$, on sera dans ce cas si la seule solution dans le cube $0\leqq x_k\leqq 1$ est tous les $x_k=1$.

Soit $A_{ik}(t)$ l'espérance mathématique du nombre de particules du type T_k, à la t-ème génération, si le processus a commencé avec une particule du type T_i. Ces expressions vérifient le système

$$x_i(t+1)=\sum_{j=1}^n a_{ij}x_j(t), \qquad i=1, \cdots, n,$$

où $a_{ij}=A_{ij}$, avec les données initiales $x_i(0)=\delta_{ik}$. Sa solution est de la forme

$$x_i(t)=\sum_{j=1}^n \varphi_{ij}(t)\lambda_j^{(t)}, \qquad i=1, \cdots, n,$$

où λ_j est racine caractéristique de $A=\|a_{ik}\|$, de multiplicité r, et $\varphi_{ij}(t)$ un polynôme de degré non supérieur à $r-1$.

A tout indice k on fait correspondre la classe $S^{(k)}=A_k\cap B_k$, avec $x_i\varepsilon A_k$ lorsque $f_i^{(t)}$ dépend de x_k pour un certain t et $x_j\varepsilon B_k$ lorsque $f_k^{(t)}$ dépend de x_j pour un certain t. Si $S^{(k)}$ est vide, T_k est dit spécial. L'ensemble de types se décompose en classes disjointes et en types spéciaux. Une classe est finale s'il existe, presque-sûrement, dans toute génération d'une particule de cette classe une particule de la même classe. L'on a, en supposant que les $f_k(x)$ possèdent des dérivées secondes finies en $(1, 1, \cdots, 1)$, le théorème suivant: pour qu'un processus soit dégénérescent, il faut et il suffit que (i) il n'y ait pas des classes finales, (ii) tous les $|\lambda_j|\leqq 1$. *M. Loève* (Berkeley, Calif.).

Kendall, David G. On the generalized "birth-and-death" process. Ann. Math. Statistics **19**, 1–15 (1948).

Consider an integral-valued Markov process $X(t)$ with $P_n(t)=\Pr\{X(t)=n\}$ satisfying the Kolmogorov equations

$$P_n'=-n(\lambda+\mu)P_n+(n-1)\lambda P_{n-1}+(n+1)\mu P_{n+1}.$$

The coefficients λ and μ measure the birth and death rates and may depend on t. The author finds a partial differential equation for $\phi(t, z)=\sum P_n(t)z^n$ and shows that $P_n(t)$ is of the form $P_n=(1-P_0)(1-\eta(t))\eta^{n-1}(t)$ for $n\geqq 1$. The probability of ultimate extinction (that is, $\lim P_0(t)$) and the median life time are determined. Next the author turns to the study of the "cumulative" population, that is, of the monotonic variable $Y(t)$ which has the same positive jumps as $X(t)$. He finds Var $(Y(t))$, Cov $(X(t), Y(t))$ and other quantities of interest. The case of periodic coefficients is treated in detail. *W. Feller* (Ithaca, N. Y.).

Pollaczek, F. Sur un problème du calcul des probabilités qui se rapporte à la téléphonie. J. Math. Pures Appl. (9) **25** (1946), 307–334 (1947).

Suppose that customers arrive in accordance with a Poisson law and are served by $s\geqq 1$ counters. If all counters are busy the customer joins a common waiting line. The servicing requires a unit time. At any time the situation is described by the number $v\leqq s$ of busy counters, the respective fractions $t_1, \cdots, t_v$ of elapsed service time ($t_r<1$) and, if $v=s$, by the number of people in the waiting line. Assuming statistical equilibrium we are interested in the corresponding probabilities $p_0, p_1(t_1), p_2(t_1, t_2), \cdots, p_s(t_1, \cdots, t_s), p_k(t_1, \cdots, t_s)$ ($k>s$). They can be obtained from geometrical considerations in form of forbidding multiple integrals which the author evaluates using complex-variable methods developed earlier for the same purpose [cf. Ann. Inst. H. Poincaré **10**, 1–55 (1946); same J. (9) **24**, 73–93 (1945); these Rev. **9**, 362; **7**, 451]. Next the author sketches an alternative approach using the continuity equations which express that statistical equilibrium exists. They form a system of partial differential equations corresponding to the Kolmogorov equation for Markov processes but involving $\partial p_v/\partial t_1, \cdots, \partial p_v/\partial t_v$. In principle, these equations can be solved by the method of generating functions, but a complete solution is obtained only for $s\leqq 3$. *W. Feller* (Ithaca, N. Y.).

Blackwell, David. A renewal theorem. Duke Math. J. 15, 145–150 (1948).

Let x_i be independent nonnegative random variables identically distributed and let $U(T)$ be the expected number of sums $s_k = x_1 + \cdots + x_k$ lying in the interval $(0, T)$. The author proves the following theorem: unless all values of x_i are integral multiples of some fixed constant, $U(T+h) - U(T) \to h/E(x_i)$ $(T \to \infty)$ for every $h > 0$. He mentions that the method used is that of Feller (and Erdös and Pollard) who obtained a corresponding result in the case excluded in the theorem stated above. *M. Loève.*

Simaika, J. B. On the significance of a typical value in the renewal theory. Skand. Aktuarietidskr. 30, 121–129 (1947).

With the usual assumptions in renewal theory let $C(t)$ be the accumulated number of elements introduced during $(0, t)$ and $P(t)$ the accumulated number of eliminated elements. H. Bolza [Berichte Physik.-Med. Gesellsch. Würzburg, N.F. 63, 97–120 (1940)] defined a quantity γ by $P(t-\gamma) = C(t)$. The author studies the asymptotic behavior of γ, its dependence on the moments of the mortality distribution, etc. *W. Feller* (Ithaca, N. Y.).

Fürth, R. On the theory of electrical fluctuations. Proc. Roy. Soc. London. Ser. A. 192, 593–615 (1948).

The author gives a general exposition of the theory of spontaneous electrical fluctuations. He criticizes the idea of a continuous harmonic spectrum of a function [apparently being unaware of the harmonic analysis of an individual function and of the corresponding harmonic analysis of stationary stochastic processes] and bases his approach on general theorems of statistical mechanics leading to appropriate forms of the equipartition principle. *J. L. Doob.*

Casal, Pierre. Statistique d'un champ homogène de vecteurs aléatoires de divergence nulle. Application à la turbulence homogène. C. R. Acad. Sci. Paris 226, 870–872 (1948).

The author assumes a random vector

$$a(x) : (a_1(x), a_2(x), a_3(x))$$

associated with each point x: (x_1, x_2, x_3), whose divergence vanishes identically, defines $A_{ij} = E\{a_i(x)a_j(x')\}$ and assumes that this expectation depends only on the difference $x' - x$. Then if $b(x) = \text{rot}\, a(x)$ and if $B_{ij} = E\{b_i(x)b_j(x')\}$, B_{ij} depends only on $x' - x$ and a differential equation is set up connecting the B's and A's. The results are then interpreted as results on the correlation functions of turbulent flow, with $a(x)$ the vector velocity. *J. L. Doob* (Urbana, Ill.).

Frenkiel, F. N. On the kinematics of turbulence. J. Aeronaut. Sci. 15, 57–64 (1948).

The author reviews the various correlation coefficients and corresponding spectral distributions that arise in the statistical theory of turbulence. He exhibits the agreement between experimental results and various correlation functions, stressing particularly correlation functions of the form $\exp(-k|r|^j)\Phi(r)$, where $j = 1$ or 2 and $\Phi(r)$ is an algebraic or trigonometric polynomial. *J. L. Doob* (Urbana, Ill.).

Mathematical Statistics

***Kelley, Truman Lee. Fundamentals of Statistics.** Harvard University Press, Cambridge, Mass., 1947. xvi+755 pp. $10.00.

This book is written as an introduction to statistics for people of limited mathematical background. The author is enthusiastic about his task and the book represents a prodigious amount of work. The discussion is not always very precise. Considerable computational details are included. The spirit of the book can be gauged from its treatment of the questions of estimation and testing hypotheses, which are the raison d'être of statistics. A detailed table of contents and a fairly good index list "estimation, pp. 332–395." This is a chapter devoted to the bivariate normal distribution and bearing the title "Estimation, regression, and correlation." The "estimation" refers to the estimates of the bivariate normal parameters, as no general discussion of estimation is given. [On pages 349 and 350 the same symbol is used for a regression coefficient and its estimate.] The index lists "hypothesis, null, p. 332, pp. 355–6." Nowhere in these pages is the rationale of testing hypotheses discussed. The references to psychological and educational practice are interesting. The reviewer omits listing a number of imprecisions he found. *J. Wolfowitz.*

Birnbaum, Z. W. On random variables with comparable peakedness. Ann. Math. Statistics 19, 76–81 (1948).

A random variable Y is said to be more peaked about a constant a than Z is about b if $P\{|Y-a| \geqq t\} \leqq P\{|Z-b| \geqq t\}$ for all $t \geqq 0$. It is shown that, if Y and Z are symmetric with density functions which are nonincreasing for positive values of the variables, and if Y is more peaked about 0 than Z, then the mean $\bar{Y}$ of a sample of size n from a population Y is more peaked about 0 than the corresponding mean $\bar{Z}$. If in addition $\bar{Y}$ is bounded, say $|\bar{Y}| \leqq a$, then $\bar{Y}$ is more peaked than the variable Z with a uniform distribution over $(-a, a)$, so that $P\{|\bar{Y}| \geqq t\} \leqq P\{|\bar{Z}| \geqq t\}$ for all t. An explicit formula is known for the latter probability.

D. Blackwell (Washington, D. C.).

van Uven, M. J. Extension of Pearson's probability distributions to two variables. III, IV. Nederl. Akad. Wetensch., Proc. 51, 41–52, 191–196 = Indagationes Math. 10, 12–23, 62–67 (1948).

The author continues his study of the bivariate Pearson system [for parts I, II cf. the same Proc. 50, 1063–1070, 1252–1264 = Indagationes Math. 9, 477–484, 578–590 (1947); these Rev. 9, 363]. In part III he determines the regions for the types, the positions of the means and of the maximum (minimum), the interrelationships among the six types, and the values of the constants for each probability function in terms of the moments. In part IV the marginal distributions are shown to consist of Pearson univariate types, and the way is cleared for the application of these results to the graduation of observed bivariate distributions.

L. A. Aroian (New York, N. Y.).

Gottschalk, V. H. Symmetrical bi-modal frequency curves. J. Franklin Inst. 245, 245–252 (1948).

The dissection of a symmetrical unimodal or bimodal probability function into two normal components is solved in two ways: (1) by moments, and (2) by moments and half moments. The sth half moment is defined as $\int_{-\infty}^{0} x^s y\,dx$, where y is a probability function. Four numerical examples

are included. No references are mentioned later than K. Pearson's paper [Philos. Trans. Roy. Soc. London. Ser. A. 185, 71–110 (1894)]. The author's solution by moments has been found previously by Charlier and Wicksell [Ark. Mat. Astr. Fys. 18, no. 6 (1924)]. The consistency and efficiency of the methods are not discussed. *L. A. Aroian.*

Sverdrup, Erling. Derivation of the Wishart distribution of the second order sample moments by straightforward integration of a multiple integral. Skand. Aktuarietidskr. 30, 151–166 (1947).

Wishart's distribution in multivariate analysis is derived by a straightforward evaluation of the n-fold integral involved. The author feels that the derivation in the bivariate case is simple enough to be included in elementary texts of statistics. *L. A. Aroian* (New York, N. Y.).

Bartlett, M. S. Multivariate analysis. Suppl. J. Roy. Statist. Soc. 9, 176–190; discussion 190–197 (1947).

An integrated account of a large part of multivariate analysis, including discriminant functions, canonical variates and the distributions of likelihood ratios and canonical roots. There are 56 references. *J. W. Tukey.*

Plackett, R. L. An exact test for the equality of variances. Biometrika 34, 311–319 (1947).

The author extends to p-variate normal universes the principle for testing the hypothesis of equal covariance matrices first developed by E. J. Pitman [Biometrika 31, 9–12 (1939); these Rev. 1, 63] and W. A. Morgan [Biometrika 31, 13–19 (1939); these Rev. 1, 64] in the univariate case. If samples, which must be of the same size for the present method, are taken from each of k p-variate normal populations, the observations on each variable in the sample are ordered at random and a set of k linear orthogonal linear forms, of which the first is proportional to the sum, is formed on the corresponding observations of the same variable from each of k universes for each of the p variables. The problem is thus reduced to that of testing the independence of two groups of variables, one of size p and the other of size $p(k-1)$. The work of Wilks and others then becomes available for the test proposed. The formal proof of the equivalence of the two tests is given and the numerical application of the test proposed is discussed and illustrated. For $k=1$, the criterion has the form of a multiple correlation coefficient and it is pointed out that in this case Fisher's work makes it possible to calculate the power of the test. *C. C. Craig* (Ann Arbor, Mich.).

Walsh, John E. On the use of the non-central t-distribution for comparing percentage points of normal populations. Ann. Math. Statistics 19, 93–94 (1948).

The results of Johnson and Welsh for the non-central t-distribution [Biometrika 31, 362–389 (1940); these Rev. 1, 346] are used to determine confidence intervals for $\theta_\alpha - \varphi_\beta$, where θ_α and φ_β are the α and β percent points for two normal populations with the same variance but unlike means. *R. L. Anderson* (Raleigh, N. C.).

Nanda, D. N. Distribution of a root of a determinantal equation. Ann. Math. Statistics 19, 47–57 (1948).

Let S' be a sample of size n' drawn from a p-variate normal population Σ' and S'' a sample of n'' drawn from a p-variate normal population Σ''. Denote the variance-covariance matrices of S' and S'' by $\|a'_{ij}\|$ and $\|a''_{ij}\|$, respectively. Let $\theta_1, \cdots, \theta_p$ be the roots of the determinantal equation $|a'_{ij} - \theta a''_{ij}| = 0$. The author determines by a new method, based on a result of P. L. Hsu [Ann. Eugenics 9, 250–258 (1939); these Rev. 1, 248] the distribution of the largest, the smallest and any intermediate root for $p=2, 3, 4, 5$, under the null hypothesis of the equality of $\|a'_{ij}\|$ and $\|a''_{ij}\|$. The results have been previously obtained by S. N. Roy [Sankhyā 7, 133–158 (1945); these Rev. 7, 317]. *L. A. Aroian* (New York, N. Y.).

Whitfield, J. W. Rank correlation between two variables, one of which is ranked, the other dichotomous. Biometrika 34, 292–296 (1947).

The author correctly conjectures formulas for the variance of Kendall's τ under three conditions: (a) when one ranked variable has no ties and the other is dichotomized, (b) when one ranked variable contains an arbitrary number of ties and the other is dichotomized, and (c) when both variables are dichotomized. Kendall verifies Whitfield's conjectures in the paper reviewed below. *S. S. Wilks.*

Kendall, M. G. The variance of τ when both rankings contain ties. Biometrika 34, 297–298 (1947).

In this note Kendall obtains the variance of his correlation coefficient τ in case there is an arbitrary number of ties in the ranks of each of the two variables from which τ is constructed. The variance is obtained for the distribution of τ over the set of pairings of all possible permutations of the ranks for each variable. Kendall shows how his general variance formula reduces (a) to results obtained by Sillitto [Biometrika 34, 36–40 (1947); these Rev. 8, 475] when one ranked variable has no ties; and (b) to three results conjectured by Whitfield [cf. the preceding review]: the case when one ranked variable has no ties and the second is dichotomized; the case when one ranked variable contains an arbitrary number of ties and the other is dichotomized; and the case when both ranked variables are dichotomized. *S. S. Wilks* (Princeton, N. J.).

Tukey, John W. Approximate weights. Ann. Math. Statistics 19, 91–92 (1948).

Let x_i $(i=1, \cdots, n)$ be independent estimates of the same mean μ and let the variance of x_i be σ^2/W_i, where $\sum W_i = 1$. The estimate of μ with minimum variance is well known to be the weighted mean $\sum W x_i$, with variance σ^2. If approximate weights $W_i(1+\lambda\theta_i)$ are used, where $|\theta_i| \leqq 1$, $0<\lambda<1$, it is proved that the variance of the approximately-weighted mean cannot exceed $\sigma^2/(1-\lambda^2)$. Hence if weights are approximated by numbers of the form $10(1)20(2)50(5)100\times 10^r$, then λ cannot exceed $1/21$, and the fractional increase in variance due to rounding errors is at most $\frac{1}{4}$ percent. The proof contains three minor misprints. *W. G. Cochran.*

Tukey, John W. Nonparametric estimation. III. Statistically equivalent blocks and multivariate tolerance regions—the discontinuous case. Ann. Math. Statistics 19, 30–39 (1948).

In the first paper of this series [Scheffé and Tukey, same Ann. 16, 187–192 (1945); these Rev. 7, 21], it was shown that the one-dimensional theory of nonparametric tolerance regions may be extended to the discontinuous case if certain equalities are replaced by inequalities. In the second paper [same Ann. 18, 529–539 (1947); these Rev. 9, 295] the theory was extended to multi-dimensional tolerance regions, in the case where the joint cumulative of the variables considered is continuous; the tolerance regions were built up of "random blocks," i.e., population space portions de-

termined by sampling. The present paper gives the extension to discontinuous distributions. Here the "cuts" between the blocks may have positive probability "coverages" and cannot be neglected; in order to get the tolerance regions sought for, one must exclude the cuts or incorporate them with the blocks in a certain manner. Rules for dealing with ties are also given. *G. Elfving* (Helsingfors).

Radhakrishna Rao, C. Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation. Proc. Cambridge Philos. Soc. **44**, 50–57 (1948).

This paper presents large sample tests based on quadratic forms in the first derivatives of the logarithm of the likelihood. No mention is made by the author that these tests are equivalent to the usual maximum likelihood tests. *G. W. Brown* (Ames, Iowa).

Roy, S. N. Notes on testing of composite hypotheses. Sankhyā **8**, 257–270 (1947).

Let $(x_1, \cdots, x_n)$ have a probability density depending on parameters $\theta_1, \cdots, \theta_{k+s}$. This paper is concerned with s-fold composite hypotheses with k degrees of freedom corresponding to the specification of the last s parameters. Neyman's sufficient conditions for the existence of similar regions [Philos. Trans. Roy. Soc. London. Ser. A. **231**, 289–337 (1933); Statist. Res. Mem. London **1**, 113–137 (1936); Ann. Math. Statistics **12**, 46–76 (1941); these Rev. **3**, 9] are discussed in some detail, leading up to Roy's broader sufficient condition, namely that similar regions exist if there exists a set of $m<n$ shared sufficient statistics for $(\theta_1, \cdots, \theta_k)$ conditional on $(\theta_{k+1}, \cdots, \theta_{k+s})$ being known. The mechanism of constructing similar regions remains the same as Neyman's. If the shared sufficient statistics are actually independent of $(\theta_{k+1}, \cdots, \theta_{k+s})$ then it is possible to construct most powerful regions against single alternatives. Alternate sufficient conditions for most powerful regions are available for $m=1$ but not for $m>1$. *G. W. Brown* (Ames, Iowa).

Wald, Abraham. Foundations of a general theory of sequential decision functions. Econometrica **15**, 279–313 (1947).

This paper deals with the problem of statistical decisions when the number of observations necessary to make a decision is not determined in advance. The problem is then formulated as follows: Let $X=\{X_i\}$ be an infinite sequence of chance variables. Any particular observation of X is given by an infinite sequence $x=\{x_i\}$. Suppose further that the probability measure function μ is not known, whereas it is known that μ belongs to a given class Ω of probability measure functions. There is, furthermore, a space D given whose elements d represent the possible decisions that can be made in the problem. Each element of D will be associated with a subset ω of Ω and making the decision d is interpreted as accepting the hypothesis that the true probability measure function μ is an element of ω. The problem is to give a rule for selecting an element d of D on the basis of the observed point x or in other words to construct a function $d(x)$, called a statistical decision function, which associates with each x an element d of D. Assume $d(x)$ to be a given decision function and $n=n(x)$ the smallest integer with the property that $d(y)=d(x)$ for any y for which $y_1=x_1, \cdots, y_n=x_n$; then $d(x)$ is determined by the first $n(x)$ coordinates and $n=n(x)$ is the number of observations needed to make a decision. With these definitions as basis there is built up a thorough theory of statistical decision functions and it is shown that this theory is intimately connected with the theory of zero-sum two-person games developed by von Neumann. *K. R. Buch* (Copenhagen).

Wald, Abraham. Asymptotic properties of the maximum likelihood estimate of an unknown parameter of a discrete stochastic process. Ann. Math. Statistics **19**, 40–46 (1948).

The author studies the asymptotic properties of maximum likelihood estimates in the case of stochastically dependent observations. Let $\{X_i\}$, $i=1, 2, \cdots$, be a sequence of chance variables. It is assumed that for any n the first n variables admit a joint probability density function $p_n(x_1, \cdots, x_n, \theta)$ involving an unknown parameter θ and the consistency relations

$$p_n(x_1, \cdots, x_n, \theta)=\int_{-\infty}^{\infty} p_{n+1}(x_1, \cdots, x_{n+1}, \theta)dx_{n+1}$$

are assumed to hold. It is shown that under certain restrictions on the joint probability distribution the maximum likelihood equation has at least one root which is a consistent estimate of θ, and any root of the maximum likelihood equation which is a consistent estimate of θ is shown to be asymptotically efficient. Therefore the consistency of the maximum likelihood estimate implies its asymptotic efficiency, since this estimate is always a root of the maximum likelihood equation. *K. R. Buch* (Copenhagen).

Lehmann, E. L. On optimum tests of composite hypotheses with one constraint. Ann. Math. Statistics **18**, 473–494 (1947).

The author deals with optimum tests of certain composite hypotheses. The results of Scheffé [same Ann. **13**, 280–293 (1942); these Rev. **4**, 107] concerning type B_1 tests are further developed and discussed. It is shown that Scheffé's theorem can be extended to cover uniformly most powerful tests against a one-sided set of alternatives. It is, moreover, shown that the method of determining explicitly the optimum test region may often be reduced to a formal procedure. Optimum tests for the composite hypothesis specifying the value of the circular serial correlation coefficient in a normal distribution are obtained. The totality of similar regions is obtained for a large class of probability laws which admit a sufficient statistic. Composite hypotheses concerning exponential and rectangular distributions are treated. *K. R. Buch* (Copenhagen).

Mosteller, Frederick. A k-sample slippage test for an extreme population. Ann. Math. Statistics **19**, 58–65 (1948).

Suppose that k continuous populations $f(x-a_1), \cdots, f(x-a_k)$, which are identical apart from translations, are given. Suppose, furthermore, that the form of these populations as well as the values of the a's are unknown. The author proposes a test for deciding whether one of the k populations has slipped to the right of the rest, i.e., whether an a_i for which $a_i>\max\{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k\}$ exists, and as null hypothesis it is assumed that all the k populations are identical. The technique is first to take samples of the same size n from the k populations, next to take the sample with the greatest observation, and in this sample to count the number r of observations greater than all observations in all the other samples. The probability of finding r or more such observations is determined and for very large n the approximate value k^{1-r} is stated. Some remarks about the power of the test and the kinds of errors by testing hypotheses are added. *K. R. Buch* (Copenhagen).

TOPOLOGY

Tutte, W. T. On the four-colour conjecture. Proc. London Math. Soc. (2) **50**, 137–149 (1948).

The author gives a slight generalization of certain results of Errera which limit the structure of the configuration which results when one pentagon is removed from a minimal irreducible regular map. *P. Franklin.*

Errera, A. Sur le problème des quatre couleurs. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 807–821 (1947).

A preliminary study of the decomposition of a regular map into concentric rings, with hints of a possible relation to the four color problem. *P. Franklin.*

Mirguet, Jean. Sur une extension de la notion d'espace topologique. C. R. Acad. Sci. Paris **226**, 772–773 (1948).

An operation of derivation on a set E is a mapping $'$ from 2^E to 2^E satisfying: $x \varepsilon S'$ if and only if $x \varepsilon (S-x)'$. Additional axioms, insuring the usual properties for the closure operator derivable from this, are considered. *R. Arens.*

Nachbin, Léopoldo. Sur les espaces uniformes ordonnés. C. R. Acad. Sci. Paris **226**, 774–775 (1948).

This note follows two others by the same author [same C. R. **226**, 381–382, 547 (1948); these Rev. **9**, 367]. The author defines, among other concepts, a "natural uniform structure" for an ordered set. Among other results, whose formulation in this review would require the reproduction of several definitions, is this: every uniformly ordered space is isomorphic to a subspace of continuous bounded functions on a suitable locally compact space. *R. Arens.*

Higman, G. The compacting of topological spaces. Quart. J. Math., Oxford Ser. **19**, 27–32 (1948).

The author shows that for a normal space S the compactification due to Gelfand and Šilov [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 25–39 (1941); these Rev. **3**, 52] coincides with that due to Wallman [Ann. of Math. (2) **39**, 112–126 (1938)] in the following strong sense: not only are the two compact extensions homeomorphic, but the homeomorphism is the identity on S. This requires demonstration of a one-to-one correspondence between the dual-ideal (in the lattice of closed subsets of S) of all closed sets in S containing a point a of S, and the ideal (in the ring of all continuous bounded real-valued functions on S) of the functions equal to zero at a. Urysohn's lemma is employed.

H. Wallman (Göteborg).

Areškin, G. Ya. Structures of locally bicompact T_1- and T_2-spaces. Doklady Akad. Nauk SSSR (N.S.) **59**, 629–630 (1948). (Russian)

The author studies locally (bi-) compact T_1- and Hausdorff spaces X by means of closed bases for X, i.e., families $\{F\}$ of closed subsets of X which contain 0 and X, are closed under the operations $\cup$ and $\cap$, and generate by arbitrary intersections all closed sets of X. Several results are announced, of which the following are typical. Theorem: a lattice S with a sublattice S' is lattice-isomorphic to a closed basis $\{F\}$ for a locally compact T_1-space X, S' corresponding to the compact sets in $\{F\}$, if and only if (1) S is a distributive lattice with 0 and 1; (2) $a' \varepsilon S'$ and $a \varepsilon S$ imply that $a' \cap a \varepsilon S'$; (3) for every $a' \varepsilon S'$, $a' \neq 0$, there exist $b \varepsilon S$ and $b' \varepsilon S'$ such that $a' \leqq b'$, $a' \cap b = 0$, and $b \cup b' = 1$. Theorem: two locally compact T_1-spaces are homeomorphic if and only if they have lattice-isomorphic closed bases.

E. Hewitt (Seattle, Wash.).

Begle, Edward G. A note on local connectivity. Bull. Amer. Math. Soc. **54**, 147–148 (1948).

It is well known that if a space S is p-LC at every point, then the statement "Every point of S has arbitrarily small neighborhoods V such that any continuous p-sphere in V bounds a continuous $(p+1)$-cell in V" holds for $p=0$ but fails, in general, for $p>0$. It is shown, however, that if a space S is LC^n, then each point of S has arbitrarily small neighborhoods V such that any continuous p-sphere, $0 \leqq p \leqq n$, in V bounds a continuous $(p+1)$-cell in $\bar{V}$. The proof given also shows that if S is p-LC (for an isolated dimension p) then the statement in quotation marks above holds true if the third V is replaced by $\bar{V}$. Throughout, "LC" may be replaced by "lc" (local connectedness in terms of compact cycles and homologies). Whether the neighborhoods V can be required to be connected is left an open question. *R. L. Wilder* (Ann Arbor, Mich.).

Coelho, R. Pereira. On the groups of certain linkages. Portugaliae Math. **6**, 57–65 (1947).

This paper is concerned with the problem of deciding when a loop (map of S^1) can be pulled away from a chain of links in R^3. There appears to be no proof in the literature even of the familiar fact (assumed, e.g., in demonstrating the properties of the Antoine point-set) that a loop which is not null-homotopic in R^3-T, where T is a solid unknotted torus, is not null-homotopic in R^3-F, where F is an endless chain of three or more links, all lying on the boundary of T. In this paper this theorem is deduced from the following lemma (slightly generalised). Let F be a closed set in R^3, $C \subseteq F$ an unknotted simple polygon and E a 2-cell embedded without singularity in R^3 as a subcomplex of a simplicial subdivision. Suppose further (condition A) that all loops in $E-F$ which are null-homotopic in R^3-F are also null-homotopic in $E-F$. Then any loop in $R^3-(F \cup E)$ which is null-homotopic in R^3-F is also null-homotopic in $R^3-(F \cup E)$. The methods of proof are combinatorial in the strict sense of the reviewer [Akad. Wetensch. Amsterdam, Proc. **29**, 611–626, 627–641 (1926)] and Alexander [Ann. of Math. (2) **31**, 292–320 (1930)]. In the application to the problem of the chain of links, C is taken to be one of the links and F the whole chain. If the links are unknotted, and are linked in the simplest possible way, condition A is satisfied, and the lemma shews that if a loop is null-homotopic in R^3-F it remains so when C is filled in by a disc. In the latter part of the paper the author, considering the group of the linkage, discusses more general dispositions of the links in which condition A is satisfied.

M. H. A. Newman (Manchester).

Scorza Dragoni, Giuseppe. Criteri per l'esistenza di punti uniti in trasformazioni topologiche del cerchio e loro applicazioni. Ann. Mat. Pura Appl. (4) **25**, 43–65 (1946).

Let G be a circular disk in the plane E and let t be a topological transformation of G into E. The author establishes a number of theorems giving sufficient conditions for the existence of fixed points for t. For example, it is sufficient that G and $\Gamma = tG$ have a common interior point and that the closure of the set of those points of g (boundary of G) which cannot be joined to infinity by arcs meeting neither g nor γ (boundary of Γ) be carried by t into the interior of G.

P. A. Smith (New York, N. Y.).

Ghezzo, Santuzza. Sulla teoria delle traiettorie di una traslazione piana generalizzata. Rend. Sem. Mat. Univ. Padova 16, 73–85 (1947).

The author establishes, by fairly direct methods, some theorems of Brouwer [Math. Ann. 72, 37–54 (1912)] about simple trajectories of topological fixed-point-free plane transformations. *P. A. Smith* (New York, N. Y.).

Chern, Shiing-shen. On the multiplication in the characteristic ring of a sphere bundle. Ann. of Math. (2) 49, 362–372 (1948).

[An outline of this paper appeared in Acad. Sinica Science Record 2, 1–5 (1947); these Rev. 9, 297.] It may be remarked that Chern's formulas (25) and (26), from which the nonimbedding theorems are proved, appear also in the work of the reviewer [Lectures in Topology, University of Michigan Press, Ann Arbor, 1941, pp. 101–141; these Rev. 3, 133]. One merit in Chern's work is the new interpretation of the terms in these formulas. *H. Whitney.*

Kaplan, Samuel. Homology properties of arbitrary subsets of Euclidean spaces. Trans. Amer. Math. Soc. 62, 248–271 (1947).

Die Homologietheorie von Čech [Fund. Math. 19, 149–183 (1932)] hat sich nur für kompakte Räume als befriedigend erwiesen. Sie wird in der vorliegenden Arbeit für beliebige separable metrische Räume R nachgebildet, indem statt endlichen unendliche offene Überdeckungen von R verwendet werden, und zwar solche, bei denen ein Punkt nur endlich vielen Elementen angehört. Es werden ganzzahlige Koeffizienten und, im Nerv der einzelnen Überdeckung, endliche Ketten zugrundegelegt; die resultierende Čech-Homologiegruppe in der Dimension k sei mit $\check{H}_k(R)$ bezeichnet. Für einen Čech-Zyklus in einer kompakten Teilmenge $A \subset R$ wird Berandung in einer (beliebigen) kompakten Teilmenge von R und Berandung in R schlechthin unterschieden; erstere ist gleichbedeutend mit Berandung des zugehörigen Vietoris-Zyklus, letztere mit dessen ϵ-Berandung für alle $\epsilon>0$ und alle homöomorphen Ummetrisierungen von R. Für eine beliebige Teilmenge $B \subset R$ wird $\check{H}_k(B)$ nicht nur in B selbst, sondern auch in R beschrieben, nämlich mit Hilfe von Überdeckungen aller Umgebungen von B in R.

Für ein (unendliches) Polyeder P stimmen die $\check{H}_k(P)$ mit den kombinatorischen Homologiegruppen von P überein. Ist B eine beliebige Teilmenge von P, C ein Čech-Zyklus in einer kompakten Teilmenge von B, so berandet C dann und nur dann in B, wenn es in jeder Umgebung von B in P eine kompakte Menge gibt, in welcher er berandet. Es wird ferner der Alexandersche Dualitätssatz mit Hilfe der $\check{H}_k$ für eine beliebige Teilmenge B der n-Sphäre S^n formuliert, mit einigen Verfeinerungen und mit Anwendungen auf die Menger-Urysohnsche Dimension von B.

B. Eckmann (Zürich).

Aleksandrov, P. A general law of duality for nonclosed sets in n-dimensional space. Doklady Akad. Nauk SSSR (N.S.) 57, 107–110 (1947). (Russian)

The duality theorem is extended to the case of an arbitrary subset A, and complement B, of a spherical S^n. Coverings of A are defined as collections of sets open in A, summing to A, each intersecting at most a finite number of the others. The system of all coverings of A gives rise to projection cycles and to factor groups $\delta^p A$, coefficients in a discrete group $\mathfrak{A}$. The system of all compacta in B generates the groups $\Delta_c^q B$ using true cycles on any compactum, modulo those homologous to zero on some compactum. Coefficients are in bicompact $\mathfrak{B}$, dual to $\mathfrak{A}$.

Let (o_α) denote the system of all open subsets of space, each of them containing A. A homology group $D^p A$, coefficients in $\mathfrak{A}$, is based on a collection of cycles $(z_\alpha{}^p)$, one from each o_α, with $z_\beta{}^p \sim z_\alpha{}^p$ if $o_\beta \subset o_\alpha$; such a cycle bounds if $z_\alpha{}^p \sim 0$ in o_α. Linking is defined for each element of $D^p A$ and $\Delta_c^q B$. Elements of $\Delta_c^q B$ for which this coefficient is zero for every element of $D^p A$ are called unlinked. These lead to a factor group $\Delta'^q B$. This is appropriately topologized, and its unique bicompactification is denoted by $\Delta^q B$ and is called the q-dimensional continuous Betti group of the set B, coefficients in $\mathfrak{B}$.

It is proved that $\delta^p A$ is isomorphic to $D^p A$. Then, through a theorem of G. Chogoshvili, it is shown that $D^p A$ and $\Delta^q B$ are dual. [There is a reference here to Chogoshvili's thesis and to a note, C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 131–132 (1945); these Rev. 7, 37. However, see also C. R. Acad. Sci. Paris 221, 15–17 (1945); these Rev. 7, 216, which seems more appropriate to this paper.] It is pointed out that this general duality theorem, using cohomology groups in B with coefficients in $\mathfrak{A}$, can be established without recourse to Chogoshvili's theorem. Finally, a theorem of Eilenberg [Bull. Amer. Math. Soc. 47, 73–75 (1941); these Rev. 2, 179] is signalized as the zero-dimensional case of the general duality established in the paper.

The central problem of this paper was also solved in recently published work by S. Kaplan [see the preceding review; the abstract for this appeared in Bull. Amer. Math. Soc. 48, 375 (1942)]. *L. Zippin* (Flushing, N. Y.).

Aleksandrov, P. General duality theorems for nonclosed sets in n-dimensional space. Mat. Sbornik N.S. 21(63), 161–232 (1947). (Russian)

This is a detailed exposition of results summarized in the paper reviewed above. An introductory section gives definitions of all relevant types of cycles and homology groups, and a discussion of the principal results. In § 1, two theorems due to Chogoshvili are proved. These concern a direct spectrum $\{Y_\alpha, \pi_\beta{}^\alpha\}$ of bicompact groups and the inverse spectrum $\{X_\alpha, \omega_\alpha{}^\beta\}$ of associated discrete character groups. The first spectrum gives a bicompact group Y, the second a discrete group X. There is defined a subgroup Y_0 of Y such that X is the group of continuous characters of the factor group $Y' = Y/Y_0$; Y is the character group of X.

The isomorphism of $D^p A$ and $\delta^p A$ [see the preceding review] is proved in § 3; the fundamental duality theorem is reduced to that proposition in § 2. The principal topic in § 4 is the zero-dimensional case (theorem of Eilenberg). Here the author constructs an example to show the distinction between the zero-dimensional homology group which results when (1) every finite point set is regarded as a zero-cycle or when (2) only even numbers of points constitute a zero-cycle.

The fifth section contains some examples to show that the subgroup $\Delta_0^q B$ consisting of "unlinked cycles" of $\Delta_c^q B$ [see the preceding review] may be nonvacuous. Some characterizations of elements of this type are established. Here a further homology group is defined, designated by $\delta_c{}^p A$, and shown to be isomorphic to $\Delta_c^p A$ when the coefficient groups are taken as discrete. This new group facilitates a study of the homomorphism of $\Delta_c^p A$ into the group $\delta_A{}^p$, defined earlier. The kernel of this homomorphism is shown to consist of those true cycles which bound in every neighborhood of

A. This leads, naturally, to a definition of homology retracts. The concluding sections are devoted to homology retracts of several types, and to a study of systems of sets constituting a "domain of duality." These latter ideas were summarized in an earlier note by the author [see the following review]. *L. Zippin* (Flushing, N. Y.).

Aleksandrov, P. Homology relations in domains of duality. Doklady Akad. Nauk SSSR (N.S.) **57**, 211–214 (1947). (Russian)

With the same notation as in other papers [see the two preceding reviews], definitions are as follows. A set $A \subset S^n$ is an r^p-set if every true cycle $z_A{}^p$ of the set A which is not compact bounding in A fails to bound in some neighborhood of A. It is a ur^p-set if this neighborhood exists independently of the cycle. Further, A is an a^p-set if to every neighborhood λ of A there may be associated a neighborhood λ' of A such that every p-dimensional cycle of the open set λ' is homologous in λ to some true cycle of the set A. The set A is a homology-retract in dimension p (coefficients in $\mathfrak{A}$) if it is ur^p and a^p. It is shown that, if A is a homology retract in dimension p (coefficients in $\mathfrak{A}$) then, with $\mathfrak{B}$ as coefficients for groups in B, $\Delta_c{}^p A$ and $\Delta_c{}^p B$ are dual.

The set A has property J^p if there is a compactum $\phi \subset A$ such that every p-dimensional true cycle of A is homologous to some true cycle of ϕ and moreover if to each $\phi_a \subset A$ there is some ϕ_b, $\phi_a \subset \phi_b \subset A$, such that every true cycle of ϕ_a bounding in A also bounds in ϕ_b. Now, for the case that $\mathfrak{A}=\mathfrak{B}$, both cyclic mod m, a two-fold homology-retract mod m is defined as a set A which is a homology retract in all dimensions and moreover has property J^p. This class of sets is closed with respect to complementation; if $A \subset S^n$ is in the class, so is $B=S^n-A$. The class includes all finite dimensional compacta of finite connectivity, mod m, and includes all "stripped" polyhedra. These comprise all sets formed by addition and subtraction of a finite number of ordinary polyhedra of arbitrary dimensions. The system of all two-fold homology retracts constitutes a domain of duality, in the sense of this paper; the construction of such a domain is the objective of the paper. *L. Zippin.*

Blakers, A. L. Some relations between homology and homotopy groups. Ann. of Math. (2) **49**, 428–461 (1948).

Pursuing a line of investigation initiated by Hopf [Comment. Math. Helv. **14**, 257–309 (1942); these Rev. **3**, 316], Eilenberg showed [Ann. of Math. (2) **46**, 480–509 (1945); these Rev. **7**, 137] that under certain conditions the homology groups of a space are determined by their homotopy groups. This raises the more general question: under what conditions is the m-dimensional homology sequence $H_m(A)\to H_m(X)\to H_m(X, A)\to H_{m-1}(A)\to\cdots\to H_0(X, A)$ of a pair (X, A) (X containing A and both arcwise connected) determined by its m-dimensional homotopy sequence $\pi_m(A)\to\pi_m(X)\to\pi_m(X, A)\to\pi_{m-1}(A)\to\cdots\to\pi_1(X, A)$? (For convenience the reviewer includes in the concept of homotopy sequence the manner in which $\pi_1(A)$ operates on the sequence.) The author finds that a sufficient condition is the vanishing of all groups of the m-dimensional homotopy sequence with the possible exception of $\pi_n(X)$, $\pi_n(X, A)$, $\pi_k(A)$ and $\pi_k(X)$, where $1\leqq k\leqq n\leqq m$, and that the $(m+1)$-dimensional homology groups are then determined modulo spherical cycles. (This result includes the results of Eilenberg [loc. cit.] as special cases.) The determination apparently does not lead to any algorithm for explicitly calculating homology groups from homotopy invariants except in very special circumstances. The author further states that, when the relative homotopy groups $\pi_i(X, A)$, $1\leqq i<n$, are all trivial, the nth relative homology group $H_n(X, A)$ may be obtained from $\pi_n(X, A)$ by making the operation of $\pi_1(A)$ on $\pi_n(X, A)$ trivial. Cap and cup products and cohomology groups are shown to fit into the general picture. *R. H. Fox* (Princeton, N. J.).

Hu, Sze-tsen. Some homotopy properties of topological groups and homogeneous spaces. Ann. of Math. (2) **49**, 67–74 (1948).

Die Klassen homotoper Abbildungen f eines Raumes X in eine topologische Gruppe G, mit $f(X_0)\subset H$ für eine Teilmenge X_0 von X und eine Untergruppe H von G, bilden bezüglich des durch die Gruppenoperation von G definierten Produktes eine Gruppe $G_H(X, X_0)$. Es werden einige Sätze über diese Gruppe, insbesondere eine hinreichende Bedingung für ihre Kommutativität, angegeben. Die n-te Homotopiegruppe $\pi_n(G)$ lässt sich nach Hurewicz und Eckmann mit $G_n(S^n, p)$ identifizieren, wo p ein Punkt der n-Sphäre S^n und e das Einselement von G ist; es wird gezeigt, dass Entsprechendes für die relativen Homotopiegruppen von G mod H gilt. Ferner wird der bekannte Satz, dass $\pi_1(G)$ kommutativ ist (allgemeiner dass G n-einfach ist im Sinne von Eilenberg, $n\geqq 1$) verallgemeinert: er gilt auch, unter natürlichen Regularitätsbedingungen, für Wirkungsräume mit zusammenhängender Isotropiegruppe. Dies wird als Hauptresultat der Note bezeichnet, ist aber ein Korollar bekannter Faserungssätze. *B. Eckmann* (Zürich).

GEOMETRY

Blumenthal, Leonard M. Congruence and superposability in elliptic space. Trans. Amer. Math. Soc. **62**, 431–451 (1947).

Two subsets A, B of a metric space are called congruent if there exists a mapping $f(A)=B$ such that dist (p, q) = dist $(f(p), f(q))$ for any two points p and q of A, superposable if there exists a congruence of A onto B which can be extended to the whole space. In this paper congruence and superposability are studied in n-dimensional elliptic space $E_{n,r}$ of radius $r>0$. In $E_{n,r}$ two phenomena may occur which are impossible in Euclidean, spherical and hyperbolic spaces: there exist congruent but not superposable subsets, either of the same or of different linear dimension, when $n>1$; not every congruence between superposable subsets of $E_{n,r}$ can be extended to the whole space. Necessary and sufficient conditions are given for the superposability of two congruent subsets P and Q of $E_{n,r}$ first in the case P and Q are finite, then in the general case. The main idea is that of introducing the spherical m-tuples $(p_1{}^*, \cdots, p_m{}^*)$ associated with an m-tuple $(p_1, \cdots, p_m)$ of $E_{n,r}$ and the corresponding matrices $(\cos(p_i{}^* p_j{}^*)/r)$. *C. Pauc* (Cape Town).

Riabouchinsky, Dimitri. Le concept de l'origine d'un nombre et les trigonométries hyperboliques. C. R. Acad. Sci. Paris **226**, 859–862 (1948).

A Minkowskian metric can be introduced into affine space by specializing a hyperboloid $x^2+y^2-z^2=K$ and calling it a sphere whose radius is space-like or time-like according as K is positive or negative. The section of such a sphere by a plane through the center $(0, 0, 0)$ is a geodesic, like a

great circle on an ordinary sphere. Taking $K=-1$ we obtain a sphere of two sheets. The upper sheet is conveniently mapped by means of polar coordinates θ and φ, such that

$$x=\sinh\theta\cos\varphi,\quad y=\sinh\theta\sin\varphi,\quad z=\cosh\theta.$$

The geodesics on this sheet provide a perfect model for the lines of the hyperbolic plane. This observation was first made by A. A. Robb [Geometry of Time and Space, Cambridge University Press, 1936, appendix].

H. S. M. Coxeter (Toronto, Ont.).

Riabouchinsky, Dimitri. Sur le problème des géométries non euclidiennes. C. R. Acad. Sci. Paris **226**, 1058–1061 (1948).

Continuing the subject of the paper reviewed above, the author remarks that two planes through a generator of the asymptotic cone $x^2+y^2-z^2=0$ cut out asymptotic geodesics representing parallel lines of the hyperbolic plane, and he deduces Lobachevsky's formula for the angle of parallelism. He also considers the sphere of one sheet $x^2+y^2-z^2=1$, whose pairs of antipodal points represent the points of the exterior-hyperbolic plane as in the reviewer's paper [Amer. Math. Monthly **50**, 217–228 (1943); these Rev. **4**, 226].

H. S. M. Coxeter (Toronto, Ont.).

Coxeter, H. S. M. A problem of collinear points. Amer. Math. Monthly **55**, 26–28 (1948).

In 1893 Sylvester proposed the following problem. Let n points of the ordinary plane have the property that the line joining any two of them passes through a third point of the set. Must the n points all lie on one line? In 1933 the question was independently proposed by P. Erdös in Hungary, where an affirmative answer was established by T. Gallai (Grünwald). In 1946 Gallai's proof was simplified by G. Hajós, using besides the notions of "point" and "line" the single relation "between." Both proofs remained unpublished, but Gallai's proof became known in America through Erdös. The same simplification of Gallai's proof was found again by the author and is published in the present paper. The paper contains also a still shorter proof by L. M. Kelly which depends on the idea of distance. *L. Fejes Tóth.*

Tuckerman, Bryant. A non-singular polyhedral Möbius band whose boundary is a triangle. Amer. Math. Monthly **55**, 309–311 (1948).

Petr, K. Sur les polygones donnés par leurs cotés qui sont inscrits dans un cercle. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 325–335 (1945).

A general method is developed for obtaining the areas and the radii of the circumcircles of all closed polygons of given sides whose vertices lie on a circle. Special cases are treated without any bibliographical references.

L. Fejes Tóth (Budapest).

Belov, N. V. Crystallographic (symmetric) methods of solving geometrical problems. Doklady Akad. Nauk SSSR (N.S.) **59**, 897–898 (1948). (Russian)

The author expresses the rotation through $2\pi/5$ about the origin in terms of oblique Cartesian coordinates with axes inclined at that angle. The result is $x'=-y$, $y'=x+Ay$, where $A=\frac{1}{2}(5^{\frac{1}{2}}-1)$. Since $A^3+A-1=0$, it is easily verified that this transformation has period 5.

H. Bedford and H. S. M. Coxeter (Toronto, Ont.).

***Maurin, J. Géométrie Descriptive à Quatre Dimensions. Premier Livre. Figures du Premier Degré. Chapitres I, II, III. Définitions, Positions, Intersections.** Gauthier-Villars, Paris, 1948. 38 pp. 200 francs.

L'idée de la géométrie descriptive de quatre dimensions a été esquissée par l'auteur [C. R. Acad. Sci. Paris **225**, 560–562 (1947); ces Rev. **9**, 198]. Le présent travail donne les détails de cette esquisse. Au premier chapitre se trouve la représentation des éléments fondamentaux (figures du premier degré): point, droite, plan et hyperplan. Les deuxième et troisième chapitres ont pour objet les parties de la géométrie descriptive qui ne font pas intervenir la métrique, c'est-à-dire les problèmes de position et les problèmes d'intersection. *O. Varga* (Debrecen).

Schwidefsky, K. Über affine Bildumwandlung durch optische Projektion. Optik **2**, 434–444 (1947).

The author discusses the use of a restitutional projector to produce an affine transformation of a given figure. The affine transformation is obtained as the product of two projective transformations so that it is necessary to prepare an auxiliary print. *E. Lukacs* (China Lake, Calif.).

Forbat, N. Sur une nouvelle méthode de projection. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 885–899 (1947).

The author discusses a bilinear mapping of space on a plane. This mapping is performed by means of a linear congruence. To find the image of a point in space the line of the congruence passing through the point is intersected with the given plane. Provisions are made to permit the determination of a spatial object from its image. Straight lines are mapped on conics in the image plane. Simple theorems of solid geometry may be transformed by this mapping into statements on families of conics in a plane.

This mapping is, however, not new. It was first proposed by L. Tuschel [Akad. Wiss. Wien, S.-B. IIa. **120**, 231–254 (1911)] and later discussed by E. Müller [Akad. Wiss. Wien, S.-B. IIa. **126**, 915–929 (1917)]. As a generalization L. Eckhart proposed mappings performed by means of a congruence of order 1 but not necessarily of class 1 [Akad. Wiss. Wien, S.-B. IIa. **132**, 177–192 (1923)]. Essentially the same mapping was used recently by Fabricius-Bjerre [Danske Vid. Selsk. Math.-Fys. Medd. **22**, no. 5 (1945); these Rev. **7**, 321]. *E. Lukacs* (China Lake, Calif.).

Sengupta, B. K. Propositions on polar reciprocals. Bull. Calcutta Math. Soc. **39**, 59–60 (1947).

Ladopoulos, P. D. Contribution to the projective geometry of the mobility of geometrical configurations. Bull. Soc. Math. Grèce **23**, 51–126 (1948). (Greek. French summary)

A plane polygon π with (different) vertices $p_1, \cdots, p_n$ consists of segments (sides) p_ip_j such that each vertex lies on at least one side. The angles of π are all the angles formed by two different sides with a common vertex. Let Δ denote a (consistent) set of changes (differentials) of the angles of π with the same order of magnitude. If one side of π is fixed and if for each choice of Δ at least one side of π undergoes a change of this same order of magnitude, then π is called stable. If a Δ exists such that each side of π (except the fixed one) undergoes a change of higher order of magnitude, then π is infinitesimally deformable. If for every choice of Δ a polygon of the same combinatorial type as π, with changed angles and the same lengths of the sides as π exists,

then π is deformable. (A given polygon need not belong to any of these three classes.) For instance, a triangle with noncollinear vertices is stable; a triangle with collinear vertices is infinitesimally deformable and a quadrilateral in the usual sense is deformable. More generally, any n-gon with less than $2n-3$ sides is deformable. By considering the cross ratios of the lines connecting one vertex to four other vertices projective criteria for stability and deformability are obtained. The following generalization of Desargues' theorem is proved. If two stable plane n-gons, in the same or in different planes, are so related that $2n-3$ pairs of corresponding sides intersect on a line, then the remaining $(n-2)(n-3)/2$ pairs of lines connecting corresponding pairs of vertices intersect on that line and the lines connecting corresponding pairs of vertices pass through one point. The whole theory is dualized to bundles.

Finally stability, infinitesimal deformability and deformability of a polyhedron P are defined in an analogous manner: Δ means a change of the dihedral angles of P by quantities of the same order of magnitude and one face is kept fixed. By associating with every edge or face of P a line or plane parallel to it through a fixed point O and intersecting the resulting configuration by a plane a polygon π is obtained. With π the polyhedron P is stable, infinitesimally deformable or deformable. Convex polyhedra in the usual sense are stable [Cauchy]. Also, some previously known examples of Bricard and others for deformable polyhedra are discussed with the present method. *H. Busemann.*

Lagrange, René. Sur les produits d'inversion. C. R. Acad. Sci. Paris **226**, 625–627 (1948).

Les notations symboliques permettent d'écrire la transformée $\bar{U}\phi$ d'une sphère ϕ de l'espace a N dimensions, par rapport à la sphère d'inversion U, sous la forme $\bar{U}\phi=\phi-2(U\phi)U$, où la scalaire $(U\phi)$ est le "produit" de deux sphères. En exprimant le produit $\overline{U_nU_{n-1}\cdots U_1}M$ des inversions par rapport aux sphères successives $U_1, U_2, \cdots, U_{n-1}, U_n$, il apparait le système de sphères $U_i=\overline{U_nU_{n-1}\cdots U_{i+1}}U_i$ associé de manière réciproque aux systèmes des sphères U_i. Le rapport (h_n/h) des masses du point transformé M_n au point initial M est exprimé par le carré d'un vecteur. *P. Belgodère* (Paris).

Casulleras, Juan. On certain quadratic transformations of space. Revista Mat. Hisp.-Amer. (4) **7**, 230–233 (1947). (Spanish)

Cette note étend à l'espace à trois dimensions un procédé de construction d'une transformation quadratique plane de Seydewitz. Soient E et E' deux espaces distincts, O et r, O_1 et r_1 un point et une droite situés respectivement dans E et E'. L'auteur envisage deux homographies ω_1, ω_2, la première faisant correspondre à la gerbe de sommet O de E la gerbe de sommet O' de E' et la deuxième faisant correspondre au faisceau de plans d'arête r de E le faisceau d'arête r_1 de E', ω_1 et ω_2 étant choisies de façon que les plans (Or) et (O_1r_1) ne soient pas homologues dans les deux homographies. La transformation étudiée est celle qui fait correspondre à un point P de E le point P' de E' où se coupent les transformés de la droite (OP) et du plan (rP). Elle est généralement biunivoque; l'auteur en détermine les éléments exceptionnels, et en établit le caractère quadratique en montrant qu'elle fait correspondre à toute droite ou tout plan générique de l'espace E une conique ou une quadrique de E' passant par O'. La transformation inverse est évidemment aussi quadratique. Le cas où l'homographie ω_2 transforme le plan (Or) en (O_1r_1) est spécialement examiné; l'auteur montre que le caractère quadratique de la transformation n'est pas détruit par l'hypothèse faite, il indique les principales particularités que cette dernière entraîne, et rattache à son étude une transformation quadratique engendrée uniquement par des homographies entre formes linéaires et signalée jadis par Crémona.
P. Vincensini (Besançon).

Turri, Tullio. Le trasformazioni quadratiche involutorie del piano associate a una stessa rete di coniche. Rend. Sem. Fac. Sci. Univ. Cagliari **16** (1946), 5–14 (1948).

Convex Domains, Integral Geometry

Arf, Cahit. Un théorème de géométrie élémentaire. Rev. Fac. Sci. Univ. Istanbul (A) **12**, 153–160 (1947). (French. Turkish summary)

Let M be a finite set of points in an oriented E^2, no three of which are collinear. A polygon with vertices in M is called quasiconvex in M if the (smallest) positive angles formed by consecutive sides of the polygon contain no points of M in their interiors. If the closed polygon $A_1A_2A_3\cdots A_s$ is quasiconvex in M then the sum of the positive angles $A_{k-1}A_kA_{k+1}$ is 2π. A rotation of the plane which carries M into itself carries each closed polygon which is quasiconvex in M into itself. *H. Busemann.*

Santaló, L. A. On the measure of the set of congruent convex figures contained in the interior of a rectangle or of a triangle. Actas Acad. Ci. Lima **10**, 103–116 (2 plates) (1947). (Spanish)

Let K be a convex figure in the plane. The position of K is determined by a point $p(x, y)$ invariably connected with K and an angle of rotation φ. The measure over a set of positions of K is defined by $\int dxdyd\varphi$. If $r(\varphi)$ denotes the area of a rectangle circumscribed to K with one pair of sides parallel to φ, then the measure of all positions of K inside a rectangle with sides a, b is $2\pi ab-2L(a+b)+\int_0^{2\pi}r(\varphi)d\varphi$, where L is the length of K. Let efg be a triangle of area A and perimeter P. Denote by $a(\varphi)$ and $p(\varphi)$ area and perimeter of a triangle similar to efg and circumscribed to K one of whose sides is parallel to φ. If each of these circumscribed triangles is smaller than efg, then the measure of all positions of K inside efg is $2\pi A+\int_0^{2\pi}a(\varphi)d\varphi-2AP^{-1}\int_0^{2\pi}p(\varphi)d\varphi$. Some applications of these theorems to geometric probabilities are given. *H. Busemann* (Los Angeles, Calif.).

Young, L. C. On the isoperimetric ratio for a harmonic surface. Proc. London Math. Soc. (2) **49**, 396–408 (1947).

There exists a positive number K with the following property. Suppose x is a harmonic surface of the type of the closed disc in Hilbert space, so that x maps the closed unit disc into the set of countable sequences of real numbers whose sum of squares is finite, and each component of x is continuous on the closed disc and harmonic on the open disc. Let d be the diameter of the range of x, let L be the length of the curve obtained by confining x to the bounding circle of its domain and let A be the area of x, which is given by the classical integral over the open disc. Then $A\leqq KLd$. *H. Federer* (Providence, R. I.).

Le Couteur, K. J. A geometrical minimum problem. Math. Gaz. **31**, 261–265 (1947).

The problem runs as follows: to find to a given polygon $A_1A_2\cdots A_n$ among the polygons $A_1'A_2'\cdots A_n'$ of given perimeter l that one which minimizes the sum $A_1A_1'+\cdots+A_nA_n'$. This is a generalization of a well-known problem of Bertrand ($l=0$, $n=3$). A necessary condition is given which determines the extremal polygon for $n=3$ and $n=4$.

L. Fejes Tóth (Budapest).

Hewes, L. I. A theory of surface cracks in mud and lava and resulting geometrical relations. Amer. J. Sci. **246**, 138–149 (1 plate) (1948).

The author proves and makes use of the following theorem, well-known to mathematicians. Among all polygons having sides of given lengths the one inscribed in a circle has maximum area. *P. Erdös* (Syracuse, N. Y.).

Fejes Tóth, L. New proof of a minimum property of the regular n-gon. Amer. Math. Monthly **54**, 589 (1947).

[In the original the author's name appears as L. F. Toth.] A very short and elementary proof of the well-known fact that the regular n-gon has the least area among all n-gons circumscribed about a given circle. *W. Fenchel.*

Fejes, László. Über das kürzeste Kurvennetz, das eine Kugeloberfläche in flächengleiche konvexe Teile zerlegt. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 349–354 (1943). (German. Hungarian summary)

Let L_n be the total length of a net of curves which divides the surface of the unit sphere into $n(\geqq 4)$ convex parts with equal areas. Then

$$L_n \geqq 6(n-2)\cos^{-1}\left(\frac{2\sqrt{3}}{3}\cos\frac{n\pi}{6n-12}\right),$$

where equality holds for the central projections of the edges of the regular tetrahedron, hexahedron and dodecahedron, and only in these cases. The proof is based on monotonicity and convexity properties (stated but not proved) of a certain elementary function. A similar argument leads to an inequality for the total length of the edges of a convex n-hedron the faces of which have equal areas. In a corollary concerning nets in the plane the author seems to use the fact that L_n is asymptotically equal to the right-hand member of the above inequality. However, this does not appear from the proof. *W. Fenchel* (Copenhagen).

Fejes, László. Über einige Extremaleigenschaften der regulären Polyeder und des gleichseitigen Dreiecksgitters. Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 51–58 (1948).

The author proves among others the following theorem. Let V_n be the volume of a polyhedron with n vertices which is inscribed in the unit sphere. Then

$$V_n \leqq \tfrac{1}{2}(n-2)\cot\frac{\frac{1}{2}n\pi}{n-2}\cdot\left\{3-\cot^2\frac{\frac{1}{2}n\pi}{n-2}\right\}.$$

The author also discusses various aspects of the isepiphane problem for polyhedra. *P. Erdös* (Syracuse, N. Y.).

Fejes Tóth, László. The isepiphan problem for n-hedra. Amer. J. Math. **70**, 174–180 (1948).

The author proves the inequality

$$F_n^3/V_n^2 \geqq 54(n-2)\tan\omega_n(4\sin^2\omega_n-1),$$
$$\omega_n=n\pi/(6n-12),$$

where F_n and V_n denote the surface area and the volume of an arbitrary convex n-hedron in ordinary space. Equality holds for the regular tetrahedron, hexahedron and dodecahedron. Hence these polyhedra are the solutions of the "isepiphanic problem": to find the n-hedron with greatest volume among all n-hedra with given surface area for $n=4, 6, 12$. These results are due to M. Goldberg [Tôhoku Math. J. **40**, 226–236 (1935)]. His argument was based on the convexity of a certain elementary function the proof of which he did not carry out. The author proves an inequality between integrals involving the minimal spherical distance of a variable point from n given points on the unit sphere and gets the above inequality as a corollary. Another corollary is a theorem of Kershner and of the author [Mat. Fiz. Lapok **50**, 40–46 (1943); these Rev. **8**, 219] concerning the density of a finite system of congruent spherical caps.

W. Fenchel (Copenhagen).

Algebraic Geometry

Zariski, Oscar. Analytical irreducibility of normal varieties. Ann. of Math. (2) **49**, 352–361 (1948).

Let $\mathfrak{o}$ be the quotient ring of a point P on an irreducible algebraic variety V. The principal result is that if V is locally normal at P (i.e., $\mathfrak{o}$ is integrally closed), then V is analytically irreducible at P (i.e., the completion $\mathfrak{o}^*$ of the local ring $\mathfrak{o}$ has no zero divisors). The question is raised as to whether also $\mathfrak{o}^*$ is integrally closed. The author gives also a new proof of a theorem of Chevalley [Trans. Amer. Math. Soc. **57**, 1–85 (1945); these Rev. **7**, 26] stating that in any case (i.e., whether or not $\mathfrak{o}$ is integrally closed) $\mathfrak{o}^*$ is always free from nilpotent elements. *I. S. Cohen.*

Châtelet, François. Sur la réalité des courbes unicursales. Revue Sci. **85**, 709–715 (1947).

M. Noether has shown [Math. Ann. **3**, 161–227, 547–580 (1871)] that any plane unicursal curve can be birationally transformed, within any field including the coefficients of its equation, into a line or conic; an arithmetical interpretation of this result was given by Hilbert and Hurwitz [Acta Math. **14**, 217–224 (1890)]. Now the author obtains an equivalent result for real unicursal curves, by using a somewhat simpler new method, which consists in first expressing the coordinates of a point describing the curve as complex rational functions of a parameter, and then determining all (generally complex) values of the parameter associated with real simple points of the curve.

B. Segre (Bologna).

Franchetta, A. Sulle curve appartenenti a una superficie generale d'ordine $n\geqq 4$ dell' S_3. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 71–78 (1947).

An algebro-geometric proof is given of a theorem stated by Noether and proved with transcendental methods by Fano and Lefschetz. It is to the effect that any curve on the general surface F^n of order $n\geqq 4$ in S_3 is the complete intersection of F^n and another surface. In view of the fact that the surfaces of a given order cut out a complete linear series on F^n, it is sufficient to show that the given curve $\bar{C}^N$ is linearly equivalent to a multiple of the plane section $\bar{C}^n$. The surface F^n is made to approach continuously the most general monoid ϕ^n; the series $|\bar{C}^N|$ goes into $|C^N|$ on the ϕ^n; and $|\bar{C}^n|$ goes into $|C^n|$. If the monoid ϕ^n is repre-

sented on a plane by projection from its $(n-1)$-ple point, the vicinity of the $(n-1)$-ple point is represented by a Γ^{n-1}; the series $|C^n|$ has on Γ^{n-1} the base points A_i, $i=1, \cdots, n(n-1)$, and thus Γ^{n-1} is fundamental for the series $|C^n|$. Replacing $\bar{C}^N$ by $\bar{C}^N+k\bar{C}^n$, so that one may suppose $\bar{C}^N$ is not contained in the canonical system of F^n, one can show that the limit system $|C^N|$ also has Γ^{n-1} virtually as fundamental curve. Were $|C^N|$ irreducible, its virtual multiplicity at the base points A_i would all be equal, and hence $|C^N|$ would be equivalent to a multiple of $|C^n|$. It is possible, however, that, as F^n approaches ϕ^n, the system $|\bar{C}^N|$ goes into a system $|C^N|$ having the $(n-1)$-ple point as base point and having some of the straight lines through this point as fixed components; in symbols: $|C^N|=|C'|+\rho\Gamma^{n-1}+\sum t_iA_i$. Replacing $|C^N|$ by $|C^N+lC^n|$, for sufficiently high l, if the term $\sum t_iA_i$ continues to make its appearance, the consequence would be that the superabundance of $|C'+lC^n|$ would be unbounded; whereas for an arbitrary $|C'|$ having the A_i as base points, it is shown that the superabundance of $|C'+lC^n|$ is at most $\sum i_m$, where i_m is the index of specialty of $|G_h+mG|$; G_h is the intersection of C' with Γ^{n-1} outside the base points and G is a group cut out by a straight line. On the other hand, the superabundance of $|C'+lC^n|$ is at least $\binom{n-1}{3}$, a limitation deduced from the fact that $|C^N|$ is a limit; and this is used to show that the term $\rho\Gamma^{n-1}$ becomes absorbed on replacing C^N by C^N+lC^n. One obtains thus that $C^N\equiv kC^n$. The virtual dimension of $|\bar{C}^N-k\bar{C}^n|$ is then calculated to be nonnegative, whence, by a theorem of Severi, one also has $\bar{C}^N\equiv k\bar{C}^n$. *A. Seidenberg.*

Gul', I. M. Singular elements of projective bundles of higher order. Doklady Akad. Nauk SSSR (N.S.) **59**, 845–846 (1948). (Russian)

Given n projectively related bundles of hyperplanes S_{n-1} in a projective space S_n (a bundle is the set of all hyperplanes through a point), any n corresponding hyperplanes of the bundles generally intersect in a point, and the locus of these points is an algebraic hypersurface G. Particular n-tuples of corresponding hyperplanes may intersect in an S_k, $k>0$, which then will lie on G, and such an S_k is referred to as a singular element of the given projective bundles. The condition that there exist a singular S_k is expressed by $k(n+1)$ homogeneous equations in $k+1$ sets of homogeneous variables. One of these sets contains n variables (the homogeneous coordinates in a bundle) and the other k sets consist each of $n-k+1$ variables. The equations are linear homogeneous in the variables of each set, and the condition is that there exist a nontrivial solution. From this it follows that if $n\geq k^2+k+1$ there always exists a singular S_k. *O. Zariski* (Cambridge, Mass.).

Berzolari, Luigi. Su un semplice problema di geometria numerativa. Boll. Un. Mat. Ital. (3) **2**, 93–95 (1947).

Determination of the number of spaces $[s]$ in $[n]$ which meet a given $[k]$ in a point, a given plane in a line, and meet N general lines; where n, s, k, N are chosen so that the problem is determinate. *J. A. Todd.*

Lage Sundet, Knut. On the geometric relation between the tangents in multiple points of algebraic curves. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 9, 29–32 (1947).

The following theorem is proved. If an n-ic curve C is just determined by points $f_1, \cdots, f_h$ of respective multiplicities $s_1, \cdots, s_h$, and some of the tangents in these points, forming $\frac{1}{2}n(n+3)$ conditions in all, then all the tangents in f_1 and f_2 and the harmonic conjugates, with respect to f_2f_3, f_2f_1 of those in f_2, touch a curve C'' determined by the following conditions: its class is $2n-s_1-s_2-s_2$; it touches the given tangents of C in f_1, f_3 and the harmonic conjugates of those in f_3; it has as multiple tangents of orders $n-s_1-s_2$, $n-s_2-s_3$, $n-s_3-s_1$, s_m, respectively, the lines f_1f_2, f_2f_3, f_3f_1, a_mb_m, where a_m is the intersection of f_1f_m with f_2f_3, b_m that of f_3f_m with f_1f_2 ($m=4, 5, \cdots, h$); if A is a given tangent to C in f_m, and c any point of the conic through $f_1f_2f_3$ and touching A in f_m, a the intersection of f_1c, f_2f_3 and b that of f_2c, f_1f_2, then the intersection of ab, a_mb_m is a point of contact of the latter with C''. The proof uses a quadratic transformation based on $f_1f_2f_3$ followed by polar reciprocation with respect to a conic having this as a self polar triangle, and C'' is just the transform of C by these two transformations together. *P. Du Val* (Istanbul).

Severi, Francesco. Teoremi di regolarità sopra una superficie algebrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 346–352 (1947).

An interesting restatement, with some extensions, of theorems on the regularity of linear systems of curves on an algebraic surface which has ordinary singularities. The emphasis is now on connected curves and one theorem states: necessary and sufficient conditions for every effective reducible curve on a surface to be connected are that all effective curves on the surface have positive virtual grade. *D. Pedoe* (London).

Nollet, Louis. Recherches sur les systèmes linéaires de courbes algébriques planes. Mém. Soc. Roy. Sci. Liége (4) **7**, 469–554 (1947).

This paper is a long and detailed account of the classification of linear systems of plane curves (with ordinary singularities) by means of the nature of the successive (pure) adjoint systems. The author is concerned with propositions valid without restriction on the genus of the curves of the system rather than with the study of linear systems of any assigned genus. It is impossible to summarise the many results in a brief compass, but we quote from the author's final theorem, which includes many of the more special results and adequately shows the nature of the work. Let $|C|$ be a linear system of irreducible curves of genus p, $|C_1|, |C_2|, \cdots, |C_h|$ its successive pure adjoint systems, with effective genera $p_1, p_2, \cdots, p_h$. Then (i) $p_h=0$ or 1; (ii) there exists a first system $|C_i|$, possibly coincident with $|C|$, for which $p_i>p_{i+1}$, and the sequence $p_i, p_{i+1}, \cdots, p_h$ is then steadily decreasing; (iii) if any one of the systems $|C_{i+k}|$ is degenerate, then either $|C_{i+k}|\equiv|C_h|$, $p_h=0$, and the system $|C_h|$ is compounded of a pencil of rational curves bisecant to the irreducible curves $|C_{h-1}|$, or else $|C_{i+k}|$ is compounded of a pencil $|E|$ of elliptic curves, $|C_h|=|E|$, $p_h=1$, and all the systems $|C_{i+k+1}|, \cdots, |C_{h-1}|$ are compounded of the pencil $|E|$, and $p_{i+k}, p_{i+k+1}, \cdots, p_h$ is an arithmetic progression with common difference one; (iv) if the systems $|C_i|, \cdots, |C_h|$ include a system $|C_j|$ of dimension greater than 2 which is not simple, then $|C_j|, |C_{j+1}|, \cdots, |C_h|$ are systems of hyperelliptic curves compounded of the same plane involution of order 2; (v) the sequence $|C_1|, \cdots, |C_h|$ can contain at most one degenerate system compounded of a pencil of genus two, this system must be $|C_1|$, and then $|C|$ is a pencil of hyperelliptic curves and $|C_2|$ is an irreducible system of hyperelliptic curves of dimension $2p-3$, compounded of the same plane involution of order 2. *J. A. Todd* (Cambridge, England).

Nollet, Louis. Sur le système adjoint à un système linéaire de courbes planes appartenant à une involution. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 556–562 (1947).

Es sei $|C|$ ein vollständiges lineares System ebener irreduzibler algebraischer Kurven vom Effektivgeschlecht p. Die Effektivdimension r des Systems sei $r>2$. Das adjungierte System vom Index 1, $|C_1|$, habe einen positiven Effektivgrad und gestatte eine ebene Involution. Ist $|C|$ einfach (d.h. gestattet $|C|$ keine Involution), so folgt eine erschöpfende Übersicht der möglichen Fälle aus Arbeiten von Castelnuovo [Mem. Soc. Ital. Sci. (3) 10, 103–123 (1894), und andere Arbeiten] und Fano [Ann. Mat. Pura Appl. (3) 24, 49–88 (1915)]. Gestattet $|C|$ eine Involution s-ter Ordnung, J, so zeigt Verf., dass nur folgende beiden Fälle möglich sind: (1) $|C|=|2D|$, wo $|D|$ ein Netz vom Effektivgrad 2 von irreduziblen Kurven eines höheren Geschlechtes als 1 ist; $|C|$ hat den Grad 8 und die Dimension 5 und besteht aus hyperelliptischen Kurven; (2) es gibt ein Büschel s-facher Sekanten, die auf der allgemeinen Kurve C eine durch J induzierte rationale Involution ausschneiden. Gestattet also $|C|$ eine Involution, die auf der allgemeinen Kurve eine irrationale Involution induziert, so ist das adjungierte System vom Index 1 notwendig einfach.

O.-H. Keller (Dresden).

Cattaneo, Nora. Sopra alcune curve covarianti delle linee piane. Comment. Math. Helv. 21, 67–72 (1948).

Es sei C_n eine ebene algebraische Kurve, P ein allgemeiner Punkt auf ihr. Verf. betrachtet die r-te und die s-te Polare von P hinsichtlich C_n, und der Ort ihrer Schnittpunkte ist die in Rede stehende covariante kurve. Verf. bestimmt ihrer Grad, untersucht di Spezialfälle $s=n-1$, weiter $r=n-2$, schliesslich $n=4$. Sie gewinnt einen neuen Beweis für die Plückersche Formel für die Anzahl der Wendepunkte.

O.-H. Keller (Dresden).

Lesieur, Léonce. Sur la rationalité et la géométrie des intersections d'hyperquadriques. Bull. Soc. Math. France 75, 113–192 (1947).

Let E_n be the n-dimensional projective space (over the field of complex numbers), Q_n an $(n-1)$-dimensional quadric in E_n, B_n the $(n-2)$-dimensional intersection of two Q_n's, C_n an $(n-1)$-dimensional cyclide, i.e., a hypersurface of degree 4, having a double Q_{n-1}. Chapter I. On a B_n there are 16 or $\infty^{2(n-4)}$ straight lines according as $n=4$ or $n>4$ [Bertini, Introduzione alla Geometria Projettiva degli Iperspazi, Messina, 1923, p. 181]. From this fact the author derives a class of birational transformations of an E_{n-2} onto itself; E_{n-2} and B_n are supposed to be in the same E_n. Let Δ and D be straight lines on B_n and P an arbitrary point in E_{n-2}. Find the intersection point A of B_n with the E_2 through Δ and P. Now if P is mapped on the intersection P' of E_{n-2} with the E_2 through D and A, then the correspondence $P\to P'$ is obviously birational.

Chapter II. Let B_n^r be the intersection of r arbitrary quadrics Q_n. As in the case $r=2$, by two rational representations of the same B_n^r a birational transformation of the parameter space onto itself is defined. Chapter III. If Q_n^1 and Q_n^2 are quadrics and if the points B and C describe Q_n^1 and Q_n^2, respectively, in such a way that the sides of the triangle ABC always pass through 3 given collinear points, then A generates a C_n. The different ways of generating a given C_n are examined. A similar theory for a cyclide of degree 3 is given.

Chapter IV. The author introduces the concept of the $n-2$ principal tangents at an arbitrary point M of an $(n-2)$-dimensional variety V_{n-2} in E_n. When M describes V_{n-2} then the principal tangents generate a V_{n-1}. If D is such a principal tangent of V_{n-2} then the tangent E_{n-1} of V_{n-1} is the same for all the points of D. Hence this E_{n-1} represents a generalization of the osculating plane of a curve in E_3. If E_{n-2} is the tangent space of V_{n-2} in M, then there exists in E_{n-2} a linear system S of quadratic cones with vertices in M, each cone being defined as the set of tangents in M of the intersection of E_{n-2} with an arbitrary V_{n-1} containing the given V_{n-2}. Now the $n-2$ straight lines through M which constitute a conjugate polar system with respect to the cones of S are the principal tangents of V_{n-2} in M. Some propositions concerning the V_{n-1} generated by the principal tangents of a B_n are dealt with.

F. J. Terpstra (Bandoeng).

Godeaux, Lucien. Sur une surface canonique d'irrégularité 1. Bull. Sci. Math. (2) 71, 7–16 (1947).

L'auteur construit, comme cas particulier de recherches précédents sur la construction de surfaces représentant des involutions appartenant aux surfaces qui représentent les couples de points non ordonnés de courbes algébriques, une surface irrégulière, d'ordre 24, de l'espace ordinaire, de genres $p_g=4$, $p_a=3$, $p^{(1)}=25$, dont les sections planes forment le système canonique. Cette surface contient un faisceau elliptique de courbes d'ordre 6 et de genre 4.

M. Piazzolla Beloch (Ferrara).

Godeaux, Lucien. Sur les droites multiples des surfaces algébriques. Bull. Soc. Roy. Sci. Liége 16, 74–77 (1947).

L'auteur étudie la singularité constituée par une droite d multiple d'ordre s pour une surface F d'ordre n, au moyen de la transformation birationnelle T obtenue en rapportant projectivement aux plans de l'espace les quadriques passant par d et osculatrices à une cubique gauche en un point O non situé sur d: le voisinage de d se transforme dans T en une quadrique Q sur laquelle est tracée la courbe γ d'ordre n qui correspond au voisinage de d sur F. L'étude de γ permet de connaître les caractères de la singularité que constitue d pour F. L'auteur donne diverses applications: il montre en particulier que si F a une droite double dont tous les points sont des points-pinces, F admet sur d des points triples, en nombre $n-4$ si le plan tangent varie le long de d, et en nombre $n-2$ lorsque ce plan reste fixe. L'auteur donne également une application au cas des droites triples.

L. Gauthier (Nancy).

Godeaux, Lucien. Sur les surfaces circonscrites à une surface cubique. Bull. Soc. Roy. Sci. Liége 16, 110–119 (1947).

If a nonruled cubic surface Φ is touched by a surface F of order $2n$ along a curve C of order $3n$ then either (i) C is the complete intersection of Φ with a surface of order n, or else (ii) Φ is a four-nodal cubic surface, and F passes through each node of Φ an odd number of times and has in addition $3n(2n-3)-4n_2+4$ double points on the curve C, where $n_2=\sum_1^4 v_i'^2$ and $2v_i'+1$ is the multiplicity of F at the node P_i of Φ $(i=1, 2, 3, 4)$.

J. A. Todd.

Godeaux, Lucien. Sur les correspondances rationnelles entre deux surfaces algébriques régulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 485–491 (1947).

L'auteur considère une involution cyclique d'ordre p, n'ayant qu'un nombre fini de points unis, appartenant à une surface algébrique régulière F. Dans le cas qu'il existe

un système linéaire appartenant à l'involution n'ayant pas pour points-base les points unis de celle-ci, l'auteur montre que l'adjoint à ce système contient p systèmes linéaires partiels appartenant à l'involution. Il en fait quelques applications notamment à la construction de systèmes linéaires qu'il appelle "pseudocanoniques" sur la surface image de l'involution. Il montre également que tous les systèmes linéaires de courbes tracées sur la surface F, et transformés en eux-mêmes par la transformation birationnelle de F en soi, génératrice de l'involution, ont le même comportement aux points unis de l'involution. *M. Piazzolla Beloch.*

Godeaux, Lucien. Construction d'une surface du cinquième ordre, de genre zéro et de bigenre un. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 492–501 (1947).

L'auteur construit un nouvel example d'une surface de genres $p_g=p_a=0$, $P_2=1$, et précisément du cinquième ordre possédant une droite double et deux points doubles (isolés) de Noether. (Il appelle "point double de Noether" un point double uniplanaire auquel est infiniment voisin un tacnode.) Pour une telle surface il montre que $p_g=p_a=0$, $P_2=1$, $P_3=2$, $p^{(1)}=1$. *M. Piazzolla Beloch* (Ferrara).

Benedicty, Mario. Sui piani doppi rappresentativi di superficie del quart'ordine. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 424–438 (1947).

By considering the properties of the branch curve of the double plane obtained by projecting a quartic surface in S_3 from a double point the author gives (i) the complete classification of quartic surfaces with isolated conical nodes which do not contain lines passing through the nodes, (ii) the complete classification of quartic surfaces containing the maximum possible number of isolated uniplanar nodes; and obtains results concerning quartic surfaces with tacnodes, in particular these which are irregular. The results, as the author points out, are not all novel but the consideration of the double plane enables him to obtain a unification and simplification of the theory.

J. A. Todd (Cambridge, England).

Chisini, Oscar, e Manara, Carlo Felice. Sulla caratterizzazione delle curve di diramazione dei piani tripli. Ann. Mat. Pura Appl. (4) **25**, 255–265 (1946).

The authors give a characterisation of the branch curves of triple planes obtained by projection of a surface of order $n(\geqq 3)$, without multiple curves, from an $(n-3)$-fold point. It is shown that such a triple plane is completely determined by its branch curve, and the equation of a projective model of such a triple plane is obtained. *J. A. Todd.*

Seifert, L. Sur les surfaces cubiques passant par deux triangles. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 77–78 (1945).

The quadric locus of biplanar nodes B_3 of the cubic surfaces passing through the two triangles $xy=0$, $zt(x+y+z+t)=0$ is $z^2+t^2+xy+xz+xt+yz+yt+zt=0$ (no proof is given). These cubics represent hyperplane sections of a hypersurface $xyu+zt(x+y+z+t)=0$ in four dimensions, and some properties of this and of its Hessian are given.

P. Du Val (Istanbul).

Seifert, L. Cubic surfaces containing two triangles in perspective relation. Rozpravy II. Třídy České Akad. **54**, no. 7, 7 pp. (1944). (Czech)

A summary is reviewed above.

Seifert, L. Les surfaces cubiques passant par une cubique plane et par trois droites situées dans un autre plan. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 79–81 (1945).

Of the cubic surfaces having two fixed plane sections, one of which consists of three lines, there are 16 with a unode U_8 and 12 with four nodes $4C_2$. The locus of binodes of surfaces of the system is a quadric. The cubic hypersurface represented by the system is briefly considered, together with its Hessian. *P. Du Val* (Istanbul).

Seifert, L. Cubic surfaces passing through a plane curve and three lines in another plane. Rozpravy II. Třídy České Akad. **54**, no. 8, 8 pp. (1944). (Czech)

A summary is reviewed above.

Seifert, L. Sur le nombre des surfaces cubiques à quatre points doubles passant par une courbe gauche du sixième degré ou par deux cubiques planes. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **45** (1944), 83–84 (1945).

The number of cubic surfaces with four nodes $4C_2$ passing through a given sextic curve (section by a quadric) is 126, 60, 26, or 8 if the curve has 1, 2, 3, or 4 double points, and 36 if it reduces to a pair of plane cubics. *P. Du Val.*

Seifert, L. Cubic surfaces with four double points passing through a skew sextic curve or two plane cubics. Rozpravy II. Třídy České Akad. **54**, no. 9, 6 pp. (1944). (Czech)

A summary is reviewed above.

Maroni, Arturo. Le serie lineari speciali sulle curve trigonali. Ann. Mat. Pura Appl. (4) **25**, 343–354 (1946).

Unter einer Trigonalkurve versteht man eine nicht hyperelliptische Kurve des Geschlechts p, auf der eine g_3^1 existiert, was für $p>4$ eine Besonderheit ist. Aus dem Normmodell einer solchen Kurve im R_{p-1} wird dann die g_3^1 durch die ∞^1 Geraden einer Normregelfläche r_2 ausgeschnitten. Die Fläche r_2 besitzt eine Minimalleitlinie C^m, die eine rationale Normkurve der Ordnung m ist. Es gilt $(p-4)/3\leqq m\leqq(p-2)/2$, und m erweist sich neben p als wichtige birationale Invariante. Es wird gezeigt, dass es zu jedem Paar m, p, das obige Ungleichungen befriedigt, Trigonalkurven gibt. Dann werden die auf einer solchen Kurve existierenden speziale Vollscharen g_n^r untersucht, speziell diejenigen, die nicht mit der g_3^1 zusammengesetzt sind. Diesen kommt noch eine weitere Konstante h zu, die die Anzahl der Gruppen von g_3^1 angibt, die einer Restgruppe der g_n^r bezüglich der kanonischen Schar angehören. Alle Werte von h mit $0\leqq h\leqq m$ sind dabei möglich. Mit Hilfe solcher Tatsachen über die g_n^r gelingt es dann, die Trigonalkurven vom Geschlecht p und von der Spezies m zu konstruieren, die dem P_r angehören und minimale Ordnung haben. Hiervon sei nur folgendes extreme Beispiel angeführt: Ebene Trigonalkurven vom geraden Geschlecht p und der Spezies $m=(p-2)/2$ haben die Minimalordnung $(p+6)/2$, einen $(p/2)$-fachen Punkt O und noch einen weiteren Doppelpunkt P. Die g_3^1 wird von den Geraden durch O ausgeschnitten. *W. Burau* (Hamburg).

Turri, Tullio. Osservazioni sulla classificazione delle superficie razionali reali. Rend. Sem. Fac. Sci. Univ. Cagliari **16** (1946), 1–4 (1948).

Stubban, John Olav. Sur les transformations quasi-involutives dans la géométrie de direction. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 11, 37–39 (1947).

A birational transformation T of the directed lines of the plane is called quasi-involutory if it satisfies the condition $T^2=J$, where J is the transformation which carries every directed line of the plane into its opposite. The object and stated results of this paper are a set of necessary and sufficient conditions for T to be quasi-involutory; but the reviewer found the reasoning obscure, as, for example, when the author says [p. 39] ". . . , c'est-à-dire, T^2 sera quasi-involutive, et, par conséquent, il en sera de même de la transformation T." *J. G. Semple* (London).

Thalberg, Olaf M. On "conic involutions." Avh. Norske Vid. Akad. Oslo. I. **1947**, no. 1, 15 pp. (1947).

The involutions here referred to are involutions of pairs of points (P, Q) of the plane such that P, Q lie always on a conic through 4 fixed points A_i $(i=1, \cdots, 4)$. Any such involution is immediately reducible to a central (de Jonquières) involution by a quadratic transformation based on three of the points A_i; but in this paper, in which the above fact is not mentioned, the theory of a certain class of conic involutions is developed directly, each involution being defined by the given set of points A_i and by a suitable coincidence curve C. The general type considered is that in which C is of order $4n+1$, with $2n$-fold points at $A_1, \cdots, A_4$, this curve being met residually by a conic K through the A_i in the double points of the involution on K, subordinate to the conic involution.

The author obtains, for arbitrary n, the associated homaloidal curve-net and investigates particularly the curves (other than K) which are self-conjugate in the involution. The cases $n=0$ and $n=1$ are discussed in detail, as well as some further cases in which the curve C is specially related to the basic quadrangle $A_1A_2A_3A_4$. *J. G. Semple.*

Villa, Mario. Sulle trasformazioni puntuali degeneri. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **9**, 19–26 (1942).

Villa, Mario. Superficie della V_4^6 di Segre e relative trasformazioni puntuali. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **9**, 133–144 (1942).

Villa, Mario. Sulle direzioni caratteristiche di una trasformazione puntuale. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **10**, 7–19 (1943).

Villa, Mario. Ricerche locali sulle transformazioni cremoniane. I. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **1**, 189–200 (1944).

Differential Geometry

Scherrer, W. Stützfunktion und Radius. I. Comment. Math. Helv. **20**, 366–381 (1947).

The author derives the fundamental equations of the theory of surfaces in ordinary space by introducing the square R of the radius vector and the square P of the function of support instead of the Cartesian coordinates, and by using a variable orthonormal trihedron determined by the normal unit vector and the tangent unit vector which is linearly dependent on the normal vector and the radius vector. Then the integrability conditions can be written in three distinct forms each stating two relations between the Beltrami operators with respect to one of the three fundamental forms applied to the functions R, P and one of the curvatures of the surface. From this result the author infers that a surface is determined if one of the fundamental forms, a suitable curvature and a strip are given, e.g., the second fundamental form, the Gauss curvature and a nonasymptotic strip. [At the points where $P=R$ the trihedron is undetermined and most of the formulae become singular. The author does not discuss this case.] *W. Fenchel.*

Câmpan, Florica. La pseudosphère de Bacaloglu. Acad. Roum. Bull. Sect. Sci. **24**, 96–105 (1943).

The curvature of Bacaloglu of a surface is defined by the expression $\frac{3}{8}(R_1^{-2}+\frac{2}{3}R_1^{-1}R_2^{-1}+R_2^{-2})$, where R_1^{-1} and R_2^{-1} are the principal normal curvatures. A surface of revolution for which this curvature is constant is called a pseudosphere of Bacaloglu. The author here determines the plane curves whose rotation generates these pseudospheres.

S. B. Jackson (College Park, Md.).

Groiss, Robert, und Kruppa, Erwin. Beiträge zur konstruktiven Flächentheorie. Akad. Wiss. Wien, S.-B. IIa. **156**, 9–48 (1948).

Let P be a given point on an analytic surface Φ. If P is the origin of a suitable coordinate system, Φ will have a Taylor expansion $z=\varphi_2(x, y)+\varphi_3(x, y)+\cdots$. Here $\varphi_\lambda(x, y)$ is a homogeneous polynomial of degree λ which is assumed not to vanish identically $(\lambda=2, 3)$. The cubic indicatrix of Φ at P is the rational cubic $\varphi_2+\varphi_3=0$. The main object of this paper is the study of Φ near P by means of this curve and of Dupin's indicatrix $\varphi_2=$constant. The methods are predominantly constructive. Most of the results deal with topics of elementary affine differential geometry such as the intersections of Φ with the planes through a given straight line, osculating quadrics, Transon's planes, etc. [cf., e.g., the fourth chapter of Blaschke's Differentialgeometrie, v. 2, Springer, Berlin, 1923]. One example may be sufficient: the Darboux tangents of Φ at P are the straight lines through P and the inflection points of the cubic indicatrix. The authors also discuss the cubic indicatrices of surfaces of revolution and of ruled surfaces, in particular of torses and quadrics. *P. Scherk* (Saskatoon, Sask.).

Myller, A. Surfaces à lignes méridiennes géodésiques. Acad. Roum. Bull. Sect. Sci. **27**, 103–104 (1947).

A meridian curve on a surface may be defined as the locus of points of the surface where the tangent line parallel to the (x, y)-plane has a given fixed direction. The author obtains a formula, involving three arbitrary functions, for those surfaces whose meridian curves are geodesics.

S. B. Jackson (College Park, Md.).

Myller, A. Surfaces à lignes méridiennes dans un faisceau de plans. Acad. Roum. Bull. Sect. Sci. **26**, 435–439 (1946).

The curve of contact of a surface with a circumscribed cylinder is called a cylindrical curve [K. Peterson, Ann. Fac. Sci. Toulouse (2) **7**, 5–43, 45–68, 69–107 (1905)]. The author determines the surfaces such that the cylindrical curves through a point are the intersections of the surface with planes through a fixed straight line. With proper

choice of coordinate axes, such a surface has an equation of form $z=ax+f[xg(y/x)]$. Some special cases are considered, but none results in any familiar simple surface.

A. B. Brown (Flushing, N. Y.).

Nadolschi, V. Généralisation projective des lignes méridiennes. Acad. Roum. Bull. Sect. Sci. **28**, 201–205 (1945).

The author defines on a differentiable surface S curves (C) which are projective generalizations of the meridians of Minding [J. Reine Angew. Math. **44**, 66–72 (1852)]. Let l denote the edge $x_1=x_4=0$ of a reference tetrahedron which is such that the plane $x_4=0$ is tangent to S at a point P_0. The projective meridians (C) of S which pass through P_0 are the curves of intersection of S with the cones which pass through P_0, circumscribe S, and have their vertices on the line l. The author studies, for generalized meridians, the question treated by A. Myller [cf. the preceding review] for cylindrical meridians. The "parallels" with respect to l are defined to be the curves (C') of intersection of S with the planes of the pencil whose axis is l. Finally the author determines the surfaces which have the property that the curves (C) relative to the edges $x_1=x_4=0$ and $x_2=x_3=0$ are contained in the pencils of planes whose axes are the edges $x_2=x_3=0$ and $x_1=x_4=0$, respectively. The two families of curves (C) thus defined form a conjugate net on S, and the well-known theorem of Koenigs concerning conjugate nets results.

P. O. Bell (Lawrence, Kan.).

Saban, Giacomo. Raccordement d'ordre élevé de deux surfaces réglées. Rev. Fac. Sci. Univ. Istanbul (A) **13**, 78–96 (1948). (French. Turkish summary)

Soient C une courbe plane quelconque, $C^{(i)}$ sa développée d'ordre i, P^i le centre de courbure de $C^{(i-1)}$ correspondant au point P de C. Si l'on considère le cercle $C_1^{(n-1)}$ de centre $P^{(n)}$ et de rayon $P^{n-1}P^n$, puis, successivement, la développante $C_1^{(n-2)}$ de $C_1^{(n-1)}$ passant par $P^{(n-2)}$, la développante $C_1^{(n-3)}$ de $C_1^{(n-2)}$ passant par $P^{(n-3)}$, $\cdots$, et finalement la développante C_1 de C_1^1 passant par P, cette dernière développante donne lieu à la même succession $P, P^1, \cdots, P^n$ que C: elle réalise une approximation d'ordre n de C au voisinage de P. Par analogie, l'auteur fait une application des propriétés de l'espace dualistique de Study à l'étude du raccordement d'ordre n de deux surfaces réglées ou de l'approximation d'ordre n d'une surface réglée au voisinage d'une de ses génératrices. Un paragraphe préliminaire rappelle les formules fondamentales de Blaschke relatives à une surface réglée quelconque R dont les génératrices sont définies par leur vecteur dualistique, introduit la courbure sphérique dualistique de R, et définit les développées successives et les développantes de R. Vient ensuite l'étude du raccordement d'ordre n de deux surfaces réglées R et R^* le long d'une génératrice commune G. Lorsqu'il en est ainsi R et R^* ont mêmes axes instantanés successifs jusqu'au $(n-1)$ième, les paramètres de distribution étant égaux ainsi que leurs dérivées par rapport à l'arc d'indicatrice sphérique jusqu'à l'ordre $n-1$. En outre les lignes de striction de R et R^* ont un contact d'ordre $n-1$ au point central de la génératrice commune G.

Le problème de l'approximation d'ordre n des surfaces réglées est rattaché à celui de l'approximation d'ordre n des courbes sphériques dualistiques. L'auteur montre que ce deuxième problème ne constitue qu'une partie du premier, ce qui tient au fait qu'une courbe sphérique dualistique est déterminée (à un mouvement polaire près dans l'espace dualistique) par son équation intrinsèque, alors qu'il n'en est pas de même pour une surface réglée.

P. Vincensini (Besançon).

Urban, Alois. Frenet formulas for nondevelopable ruled surfaces. Rozpravy II. Třídy České Akad. **52**, no. 18, 20 pp. (1942). (Czech)

A German summary appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **43**, 203–205 (1942); these Rev. **8**, 488.

***Schneidt, Max. Flächentransformationen durch zyklische Strahlensysteme.** Wissenschaftliche Beilage zum Bericht der Städtischen Oberschule für Mädchen am St. Anna-Platz in München für das Schuljahr 1946/47, 31 pp.

If the lines l of a congruence L are in one-to-one correspondence with the points of a surface S, each line lying in the tangent plane at the corresponding point P of S, and the focal points of each line l lie on the tangents to the lines of curvature of S at P, then L is said to be cyclic. A cyclic congruence is called a K-congruence if it is conjugate to the lines of curvature of a surface Σ; this surface Σ is said to be a cyclogic surface (zycloge Fläche). A congruence is said to be isothermal if its developables intersect two isothermic surfaces in their lines of curvature. These ideas are applied in discussing the theory of transformations of surfaces. Some typical theorems may be stated as follows. A cyclic normal surface consists of the normals to that surface which has the same spherical representation as a surface with constant curvature. The two generating surfaces of an isothermal congruence cut each line of the congruence in points harmonic to the focal points on the line. The developables of an isothermal congruence cut its middle surface in an orthogonal net. Applications are also made to congruences of Guichard and of Ribaucour.

V. G. Grove (East Lansing, Mich.).

Maxia, A. Geometria proiettiva differenziale dei 2-tessuti in S_3. Rend. Circ. Mat. Palermo **63**, 93–110 (1942).

After some preliminaries the author considers three invariant cones $\gamma_1, \gamma_2, \gamma_3$ important for the study of complexes and he fixes a local system of reference for the neighborhood of third order of a point of a 2-net. He deduces the conditions of applicability of a 2-net of second order on one of third order and then he determines the geometrical significance of the invariants found. The complexes $[t_1]$, $[t_2]$ and $[t_3]$ formed by means of the tangents of the curves and the sections of the osculating planes are then investigated. Up to the first order a projectivity of Chasles is used, well-known from the theory of ruled surfaces. The same line of thought is then followed with the introduction of different anholonomic congruences connecting the 2-net in an invariant way with the complexes. Among these the most interesting are those built from the straight lines t_1, t_2, t_3 associated with the points of one of the sets of directrices σ_{12}, σ_{13}, σ_{23}. Necessary and sufficient conditions are found for these sets to be holonomic, that is, composed of ∞^1 ordinary congruences. Finally a result is given concerning special 2-nets and generalizing former results on double systems of straight lines on a surface.

J. A. Schouten (Epe).

Blank, Ya. P. Nets conjugate in the sense of Clifford. Doklady Akad. Nauk SSSR (N.S.) **59**, 1231–1234 (1948). (Russian)

Dans l'espace elliptique nous établissons entre les tangentes en un point d'une surface S deux correspondances

projectives: la tangente t_2 est dite conjuguée à droite (à gauche) au sens de Clifford (en abrégé conjugée C-D (ou C-G)) de la tangente t_1, si elle touche la courbe de contact de la surface S avec la réglée engendrée par les tangentes de S parallèlles à droite (à gauche) à t_1, au sens de Clifford; par suite t_1 est conjuguée C-G (ou C-D) de t_2. La relation analytique pour qu'il en soit ainsi est

$$b_{ik}du^i\delta u^k = \pm g^{\frac{1}{2}}(du^1\delta u^2 - du^2\delta u^1)$$

où g_{ik}, b_{ik} sont le premier et le second tenseur de l'espace elliptique, le rayon de courbure de cet espace étant pris comme unité de longueur. La relation précédente peut encore s'écrire $b_{ik}p^iq^k = \pm \sin\omega$, où p, q sont les vecteurs unité des tangentes conjuguées et ω leur angle. Les rayons doubles de ces homographies sont les tangentes asymptotiques. Pour une surface de courbure nulle, ces homographies dégénèrent. Si t_i $(i=1, 2)$ sont conjuguées-C et si t_i' est conjuguée de t_i au sens de Dupin, t_1' et t_2' sont conjuguées-C.

Envisageons une déformation de la surface S avec conservation d'un réseau conjugué que l'on prendra pour réseau de coordonnées; en posant $b_{ik}=g^{\frac{1}{2}}\Delta_{ik}$, la condition pour que le réseau de coordonnées soit conjugué-C s'écrit $\Delta_{12}=\pm 1$ et alors la courbure gaussienne est $K=\Delta_{11}\Delta_{22}$; posant $\Delta_{11}=tK^{\frac{1}{2}}$, $\Delta_{22}=t^{-1}K^{\frac{1}{2}}$, les formules de Codazzi donnent

$$\frac{\partial \log t}{\partial u} = at^2+bt+c, \quad -\frac{\partial \log t}{\partial v} = a_1t^{-2}+b_1t^{-1}+c_1,$$

en posant

$$a=\Gamma^1_{22}, \quad b=-2\Gamma^1_{12}K^{-\frac{1}{2}}, \quad c=\Gamma^1_{11}+\tfrac{1}{2}(\log K)_u,$$
$$a_1=\Gamma^2_{11}, \quad b_1=-2\Gamma^2_{12}K^{-\frac{1}{2}}, \quad c_1=\Gamma^2_{22}+\tfrac{1}{2}(\log K)_v.$$

La condition de compatibilité du système en t est une équation algébrique en t de degré 4: si donc le réseau reste conjugué sur cinq surfaces isométriques nouvelles, la surface S admet une déformation continue où ce réseau reste conjugué-C. La condition nécessaire et suffisante pour que cela arrive est exprimée par $\cos\omega = UV$, où ω est l'angle des lignes de coordonnées, U une fonction de u, V une fonction de v.

Quand le réseau conjugué est formé de courbes minima, la courbure moyenne de la surface est égale à i; une telle surface admet une déformation continue où ce réseau reste conjugué; son ds^2 est $(1+uv)^2UVdudv$, de sorte qu'elle est isométrique à une surface de l'espace euclidien, laquelle peut se déformer d'une façon continue de sorte que le réseau des lignes minima reste conjugué. Cette propriété se rattache au théorème suivant: à chaque couple de surfaces applicables de l'espace euclidien correspond un couple de surfaces applicables de l'espace elliptique.

Dans cette même note, l'auteur utilise une représentation de la surface dans l'espace des quaternions pour le cas d'une déformation continue avec réseau conjugué-C permanent au cours de la deformation. *B. Gambier* (Paris).

Vranceanu, G. Sur les invariants de l'équation de Laplace. Acad. Roum. Bull. Sect. Sci. **23**, 492–496 (1942).

The equation of Laplace $\partial^2x/\partial u\partial v + a\partial x/\partial u + b\partial x/\partial v + cx = 0$, where a, b, c are functions of u, v, is fundamental in the projective theory of linear nets. A certain pair of absolute invariants associated with this equation have been used by Tzitzeica in his geometric investigations [C. R. Acad. Sci. Paris **200**, 191–192 (1935)]. The present author came to study these same invariants in connection with the equivalence theory of the equation of Laplace [Bul. Fac. Şti. Cernăuţi **12**, 167–192 (1938)]. The present paper restates and extends these previous results. *J. L. Vanderslice* (College Park, Md.).

Marcus, F. Sur quelques résultats de M. H. Ionas et G. Tzitzeica. Bull. Math. Soc. Roumaine Sci. **47**, 29–34 (1946).

This paper adds some further properties and gives new proofs of others of periodic sequences of Laplace transforms of a conjugate net of period four in which the congruences of the diagonals of the skew quadrilateral are W-congruences. Among these new properties the following may be chosen as representative. If the developables of the W-congruences generated by the diagonals correspond, then one of the four vertices describes a net with equal point invariants, and conversely if in a periodic sequence of Laplace of period four one of the vertices describes a net with equal point invariants, then the developables of the congruences of diagonals correspond. *V. G. Grove.*

Charrueau, André. Sur une transformation de contact. C. R. Acad. Sci. Paris **225**, 1262–1264 (1947).

L'élément de contact (A, ω) aux coordonnées x, y, z, p, q et son transformé (A', ω') aux coordonnées x', y', z', p', q' soient liés par les équations

$$x' = -\alpha_1 py - \lambda', \quad y' = \alpha_1(-z+px) - \mu', \quad z' = \alpha_1(y+\nu p) - \nu',$$
$$p' = -(x+\nu q)/(px+qy), \quad q' = (-y+\nu p)/(px+qy)$$

(α_1, λ', ν', μ', ν sont donnés), qui définissent la transformation de contact T. L'auteur étudie les divers cas de T et les propriétées géométriques des objets, spécialement d'un complexe linéaire des droites, qui sont étroitement liés avec T. La définition géométrique de la transformation T a été établie par l'auteur [C. R. Acad. Sci. Paris **225**, 1055–1058 (1947); ces Rev. **9**, 305]. *F. Vyčichlo* (Prague).

Charrueau, André. Sur une transformation de contact. C. R. Acad. Sci. Paris **226**, 155–157 (1948).

L'auteur étudie des propriétées géometriques des objets liés avec la transformation de contact T définie dans ses notes antérieures [voir l'analyse ci-dessus]. Les éléments qui définissent la transformation sont spécialisés. *F. Vyčichlo* (Prague).

Havlíček, Karel. Developable surfaces in differential line geometry. Rozpravy II. Třídy České Akad. **53**, no. 42, 24 pp. (1943). (Czech)

A German summary appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 581–583 (1943); these Rev. **8**, 488.

Vyčichlo, F. On the equipollent connection in the Möbius plane geometry. Rozpravy II. Třídy České Akad. **53**, no. 28, 14 pp. (1943). (Czech)

A German summary appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 367–376 (1943); these Rev. **8**, 532.

Selabasso, Maria. I complessi lineari involutori. Rend. Sem. Mat. Univ. Padova **16**, 159–211 (1947).

Après avoir rappelé la définition de la grassmannienne des S_k de S_n et celle des formes différentielles extérieures, l'auteur associe, selon une idée de A. Comessatti, à tout complexe linéaire de S_k, défini comme ensemble des S_k dont les coordonnées grassmanniennes sont liées par une relation linéaire, une forme différentielle extérieure à coefficients

constants, de degré $k+1$, définie à une facteur constant près. On appelle alors produit de complexes le complexe dont la forme associée est le produit des formes associées aux complexes facteurs: ce produit est associatif et commutatif et a une signification projective intrinsèque. On dit que des complexes sont en involution lorsque leur produit est le "complexe identique," défini formellement comme complexe dont la forme associée à tous ses coefficients nuls. Deux complexes sont en involution normale si, de plus, leurs espaces générateurs ont des dimensions duales. L'auteur étudie ensuite complètement la structure projective des complexes en involution dans S_1, S_2, S_3, S_4, S_5.

L. Gauthier (Nancy).

van der Woude, W. On conformal differential geometry. Theory of plane curves. Nederl. Akad. Wetensch., Proc. 51, 16–24 = Indagationes Math. 10, 3–11 (1948).

This paper is an attempt to improve on previous efforts by various writers at initiating a conformal theory of plane curves. The essential point is the use of a more natural local tetracyclic reference system for the setting up of Frenet equations, one having the osculating circle as first axis. The author includes a comparison of his method and that of Haantjes and Smits [Nieuw Arch. Wiskunde (2) 22, 34–47 (1943); these Rev. 7, 327]. *J. L. Vanderslice.*

Norden, A. Surfaces of zero curvature in a biaxial space. Doklady Akad. Nauk SSSR (N.S.) 58, 1597–1599 (1947). (Russian)

The space K is a four-dimensional space whose group of motions is isomorphic to the subgroup of the projective group which leaves invariant two conjugate imaginary lines, the absolute lines of the space. By choosing a canonical tetrahedron of reference the coordinates of conjugate points are $\bar{x}^1=-x^3$, $\bar{x}^2=x^1$, $\bar{x}^3=-x^4$, $\bar{x}^4=x^3$. The author maps the space K on a complex affine plane C so that to a point of K corresponds the point $X=x^1+x^2i$, $Y=x^3+x^4i$ of the plane C. To motions in K correspond central affine transformations in C. If $\mathbf{R}=\mathbf{R}(w)$, where the parameter w is chosen so that $XdY/dw-YdX/dw=$constant, is the equation of an analytic curve in C, then $d^2\mathbf{R}/dw^2+k\mathbf{R}=0$, where k and w are constant multiples of the radial curvature and radial arc of the curve. If $z=Y/X$ is the angular coefficient then $w=f(z)$ so that $X=\sqrt{f'(z)}$, $Y=z\sqrt{f'(z)}$ and $k=\frac{1}{2}\{z,w\}$, where $\{z,w\}$ is the Schwarzian derivative. The surface in K corresponding to such a curve is given by $x^a=z^a\cos\theta+\bar{z}^a\sin\theta$, where $x=\sqrt{f'(z)}=\rho e^{i\theta}$, $z=x+iy$ and $z^1=1$, $z^2=0$, $z^3=x$, $z^4=y$. It follows that this surface has curvature zero so that its intrinsic geometry is Euclidean.

M. S. Knebelman (Pullman, Wash.).

Nicolescu, Alex. V. Sur la géométrie centroaxiale différentielle des courbes planes et gauches. Bull. Math. Soc. Roumaine Sci. 47, 66–143 (1946).

By centroaxial parabolic geometry (in the plane) is meant the geometry of the group G_5 of projective transformations leaving invariant a point and a straight line through it. The case when the point is not on the line has been thoroughly studied by Mayer and Myller [Ann. Sci. Univ. Jassy 18, 234–280 (1933)]. Popa investigated a certain subgroup G_4 of G_5 [thesis, Jassy, 1931]. In part I of the present paper the author studies the geometry of plane curves under the full group G_5 using Halphen's method of reduced equations and Cartan's method of moving reference systems. Some representative topics are: canonical form of a curve in neighborhood of a point, Frenet equations, curvature and its geometric meaning, analogue of the Halphen point, duality, G_5-arc length and its metric interpretation. This last topic leads to a new projective invariant of any two points of a plane curve involving elements of the second order. In part II the line of development followed in part I is repeated for the analogous space geometry of the group G_9 leaving invariant a point, line and plane, all in incidence. Here, however, certain covariant quadrics play important roles. Metric interpretations of invariants are particularly interesting especially as applied to certain classes of curves characterized by specific values of the first curvature.

J. L. Vanderslice (College Park, Md.).

Min, Szu-Hoa. On a generalized hyperbolic geometry. J. London Math. Soc. 22, 153–160 (1947).

At every general point A of a surface in ordinary 3-dimensional projective space a frame of reference can be chosen such that $z=xy$ is the equation of the Lie quadric at A. It is proved that the equation of the surface then has the form

$$Z=xy+\tfrac{1}{6}(\alpha x^3+\delta y^3)+\tfrac{1}{24}(ax^4+4bx^3y+3\alpha\delta x^2y^2+4dxy^3+ey^4)+\cdots.$$

The whole family of frames depends on six parameters. From this family an arbitrary two-parameter family can be chosen such that at each point of the surface there is one frame of this family. By means of this family to each point of space a ds^2 can be found. It is proved that this ds^2 is independent of the choice of the two-parameter family if and only if the surface is a quadric. To get a definition of ds^2 intrinsically related to a (nonruled) surface it is necessary to consider the frames of reference of the fourth order attached to the surface. This ds^2 depends on the six fundamental invariants of the surface. To this ds^2 the pseudo-parallelism and the curvature affinor are constructed. Cartan notation is used. *J. A. Schouten* (Epe).

Urban, A. On the geodesic representation between two-dimensional Riemannian spaces. Nederl. Akad. Wetensch., Proc. 51, 269–279 = Indagationes Math. 10, 95–105 (1948).

Let $\mathfrak{E}^{\lambda\mu}$ be the bivector density of weight $+1$ with components $\mathfrak{E}^{11}=\mathfrak{E}^{22}=0$, $\mathfrak{E}^{12}=-\mathfrak{E}^{21}=1$. Let $a_{\lambda\mu}$ be the fundamental tensor of a V_2 and let K_μ be the first derivative of the Gaussian curvature K. Then the scalar density of weight 5, $\mathfrak{S}=\mathfrak{a}^2K^\sigma(\nabla_\mu K_\lambda)K_\nu\mathfrak{E}^{\lambda\nu}$, where $\mathfrak{a}=\det\|a_{\mu\nu}\|$, is shown to be a geodesic invariant. A necessary and sufficient condition that $\mathfrak{S}$ shall vanish at all points of a Riemannian space V_2 with $K\neq$constant is that the orthogonal trajectories of the curves $K=$constant shall be geodesics. The quantities ${}^*\Gamma^\nu_{\lambda\mu}=\Gamma^\nu_{\lambda\mu}+2A^\nu_\lambda P_\mu)$, where $P_\mu=-\frac{1}{4}\partial_\mu\log\mathfrak{a}+\frac{1}{8}\partial_\mu\log\mathfrak{S}$, are the coefficients of a connection which is geodesically invariant, and a necessary and sufficient condition that ${}^*\Gamma^\nu_{\lambda\mu}$ shall be Riemannian and not integrable is given. When ${}^*\Gamma^\nu_{\lambda\mu}$ is Riemannian and not integrable, then for the fundamental tensor ${}^*a_{\lambda\mu}$ we have ${}^*a_{\lambda\mu}=c\mathfrak{b}^{-1}\mathfrak{S}^{-1}b_{\lambda\mu}$, where $b_{\lambda\mu}=Ka_{\lambda\mu}+P_\lambda P_\mu-\nabla_\lambda P_\mu$, $\mathfrak{b}=\det\|b_{\lambda\mu}\|$, and $0\neq c=$constant. *A. Schwartz.*

Haimovici, M. La géométrie intrinsèque des variétés Riemanniennes non holonomes. J. Math. Pures Appl. (9) 25 (1946), 291–305 (1947).

Le but du présent travail est de construire la géométrie intrinsèque des variétés Riemanniennes non holonomes. La pensée, qui forme la clef de voute de cette étude consiste en la réduction de la géométrie intrinsèque à la géométrie semi-intrinsèque bien connue à l'aide d'hypothèses appropriées. Analytiquement cette réduction est la suivante: la

géométrie intrinsèque est déterminée par un certain groupe linéaire de formes de Pfaff, dont une partie définit la variété, l'autre la métrique. La géométrie semi-intrinsèque est aussi déterminée par un groupe linéaire de formes de Pfaff. Mais dans ce groupe une partie des formes de Pfaff définit un élément plan normal à chaque point de la variété. Le problème à resoudre est de réduire le groupe appartenant à la géométrie intrinsèque au groupe de la géométrie semi-intrinsèque. Géométriquement cela signifie naturellement la construction d'un élément plan normal à chaque point de la variété. L'auteur emploie la méthode d'É. Cartan des formes de Pfaff bien connue dans la théorie des groupes continus et dans la géométrie différentielle. *O. Varga.*

Petrescu, St. Sur les hypersurfaces non holonomes V^{2p}_{2p+1}. Acad. Roum. Bull. Sect. Sci. **25**, 249–253 (1943).

This note presents some rather technical results in the equivalence theory of nonholonomic metric hypersurfaces in $2p+1$ dimensions. A set of p invariants I_α are exhibited whose constancy implies that the group of automorphisms contains a maximum of $3p+1$ parameters, so-called spaces of constant curvature. Among other results is a necessary and sufficient condition that a nonholonomic hypersurface V^{2p}_{2p+1} have specific curvature constants I_α.

J. L. Vanderslice (College Park, Md.).

Petrescu, St. Sur les hypersurfaces non holonomes V^{2p+1}_{2p+2}. Acad. Roum. Bull. Sect. Sci. **25**, 313–317 (1943).

The investigation of an earlier paper [see the preceding review] is repeated with certain modifications of procedure and results for the case of a nonholonomic hypersurface V^{2p+1}_{2p+2} in even-dimensional space. Noteworthy is the exhibition of two types of always inequivalent spaces of constant curvature. *J. L. Vanderslice* (College Park, Md.).

Egorov, I. P. On the order of the group of motions of spaces with affine connection. Doklady Akad. Nauk SSSR (N.S.) **57**, 867–870 (1947). (Russian)

Soient x^i, $i=1, \cdots, n$, les coordonnées du point de l'espace à n dimensions doué d'une connexion affine avec les objets $\Gamma^\alpha_{\beta\gamma}=\Gamma^\alpha_{\gamma\beta}$, $\alpha, \beta, \gamma=1, \cdots, n$. L'opérateur infinitésimal (1) $Xf=v^\sigma p_\sigma$, $\sigma=1, \cdots, n$, où $p_\sigma=\partial F(x^1, \cdots, x^n)/\partial x^\sigma$, $v^\sigma=v^\sigma(x^1, \cdots, x^n)$ définit des mouvements si l'on peut écrire (2) $D\Gamma^\alpha_{\beta\gamma}=dv^\alpha_{\beta\gamma}-R^\alpha_{\beta\gamma\sigma}v^\sigma=0$, où $R^\alpha_{\beta\gamma\sigma}$ est le tenseur de courbure et le symbole d appartient à la dérivée covariante avec $\Gamma^\alpha_{\beta\gamma}$, D est le symbole de la dérivée de Lie dans le champ v^σ qui détermine la transformation infinitésimale. L'ensemble des r opérateurs linéairement indépendents (1) qui satisfont à (2) et forment le groupe de Lie à l'ordre maximum s'appelle le groupe complet des mouvements de l'espace donné.

L'auteur pose la question s'il existe dans les espaces affines à n dimensions des groupes de Lie d'ordre n^2+n-1, qui sont des groupes complets des mouvements. Jusqu'aujourd'hui le maximum pour l'ordre des groupes était le nombre $r=n^2+n$. Pour la résolution du problème l'auteur établit quelques lemmes et ensuite le théorème: l'ordre des groupes complets des mouvements dans les espaces à n dimensions avec une connexion affine et qui sont différents des espaces affine-euclidiens est moindre ou égal à n^2 pour $n\geqq 3$. [Pour $n=2$ le théorème est aussi valable.]

F. Vyčichlo (Prague).

Su, Buchin. A characteristic property of affine collineations in a space of K-spreads. Bull. Amer. Math. Soc. **54**, 136–138 (1948).

In this note the following theorem is proved. A necessary and sufficient condition that a mapping of a space of K-spreads upon itself shall satisfy in order that the covariant differentiation of a tensor be interchangeable with the Lie derivative is that the mapping be an affine collineation of the space. *E. T. Davies* (Southampton).

Pickert, Günter. Elementare Behandlung des Helmholtzschen Raumproblems. Z. Angew. Math. Mech. **25/27**, 136–137 (1947).

If the following conditions are satisfied: (I) $\mathfrak{G}$ is a subgroup of the linear homogeneous group in E_n; (II) if $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, $\mathfrak{d}$ are linearly independent vectors of E_n, there exists a transformation R of $\mathfrak{G}$ such that $R\mathfrak{a}=r\mathfrak{c}$ $(r>0)$; $R\mathfrak{b}=s\mathfrak{c}+t\mathfrak{d}$ $(t>0)$; (III) if $\mathfrak{a}$, $\mathfrak{b}$ are linearly independent vectors of E_n and if $R\mathfrak{a}=a\mathfrak{a}$ $(a>0)$; $R\mathfrak{b}=c\mathfrak{a}+d\mathfrak{b}$ $(d>0)$, for an R of $\mathfrak{G}$, it follows that R is the identical transformation, then it is proved for $n=3$ that $\mathfrak{G}$ is the orthogonal group with respect to a positive definite fundamental tensor. For $n=2$ the proposition is not true. For $n\geqq 4$ two other conditions have to be introduced instead of (II).

J. A. Schouten (Epe).

Petiau, Gérard. Sur les relations tensorielles entre densités de valeurs moyennes en théorie de l'électron de Dirac. I. J. Math. Pures Appl. (9) **25** (1946), 335–346 (1947).

The author gives what is claimed to be a simple derivation of the well-known quadratic relations between the tensor quantities associated with a four component spinor. The tensor quantities referred to are a scalar, a vector (the current vector), an anti-symmetric tensor, a pseudo-vector, and a pseudo-scalar. *A. H. Taub* (Princeton, N. J.).

NUMERICAL AND GRAPHICAL METHODS

***Proceedings of a Symposium on Large-Scale Digital Calculating Machinery.** The Annals of the Computation Laboratory of Harvard University. Vol. XVI. Harvard University Press, Cambridge, Mass., 1948. xxix+302 pp. $10.00.

This book gives a comprehensive survey of the state of knowledge about automatic digital computing machines at the time of the symposium (January, 1947). Included in it are accounts of work carried out during recent years and hitherto not publicly available. The work of C. Babbage on this subject about a century ago is discussed by R. H. Babbage. In addition to descriptions of the two Harvard machines (mark I and mark II), the ENIAC and the Bell relay computer, there is a considerable amount of material on the research and development work on components for the input and output of such machines and for their memories. There are general papers by S. H. Caldwell, L. Couffignal and A. T. Waterman.

Several papers are concerned with problems which have been attacked by automatic digital computing machines or which appear amenable to such treatment. R. J. Seeger discusses two explosion problems. One has been handled in several ways, by H. Polachek, A. N. Lowan and R. M. Bloch. The other was treated using a procedure suggested

by J. von Neumann: instead of replacing, as usual, the relevant partial differential equation by a system of finite difference equations the continuous model was replaced by a discrete "quasi-molecular" one and a set of algebraic equations obtained when the time derivatives in the equations of motion of the particles were replaced by differences. H. W. Emmons discusses problems in turbulent flow. L. S. Dederick suggests an improvement in the Heun method of step-by-step solution of differential equations and discusses some of the devices used to overcome some of the limitations of the ENIAC. Some problems in economics for which automatic digital computing machines appear essential are discussed by L. W. Leontief.

A. W. Wundheiler underlines the fundamental importance of error estimation in computational processes used in automatic digital computing machines. Crude estimates, by the addition of maximum errors, are generally unacceptable and so some probabilistic approach appears essential [cf. J. von Neumann and H. H. Goldstine, Bull. Amer. Math. Soc. **53**, 1021–1099 (1947); these Rev. **9**, 471]. R. Courant indicates specific physical problems giving rise to partial differential equations and suggests possible methods of handling them numerically. He points out important open questions in what may be called modern numerical analysis.

H. Rademacher examines the effect of the accumulation of rounding-off and truncation errors in some quadrature processes. By using smaller steps the error in the basic formulae is reduced but at the same time a larger number of steps is required to cover a given range and so there is a possibility of a serious accumulation of rounding-off errors.

The preparation of problems for EDVAC-type machines is discussed by J. W. Mauchly and for the Harvard mark I machine by J. O. Harrison, Jr. *J. Todd* (London).

***Tables for the Design of Missiles. By the Staff of the Computation Laboratory.** Harvard University Press, Cambridge, Mass., 1948. lv+226 pp. $9.00.

This volume gives tables of auxiliary functions which are needed in calculations of volume, position of mass centre and moments of inertia of projectiles in the form of solids of revolution obtained by rotation of a plane figure, with boundary composed of straight lines or arcs of circles, about an axis in its plane. The solid may be considered as made of elementary parts of four kinds: (1) right circular cylinders, (2) frustums of right circular cones, (3) ogives and (4) solids generated by the revolution of a fillet. An ogive is generated by revolution of a circular arc about a chord of the circle (not cutting, though it may meet, the arc). A fillet is an area bounded by an arc of a circle less than a right angle and the tangents at its extremities.

Tables I to IV are concerned with ogives: (I) H_1, H_2, H_3, H_4, each with Δ, 6 decimals, $a=\sin\phi=0(0.001)0.999$, where

$$H_1=20a^{-5}\{2a-\sin^{-1}a-a(1-a^2)^{\frac{1}{2}}-\tfrac{1}{3}a^3\},$$
$$H_2=24a^{-6}\{a^2-\tfrac{1}{4}a^4-\tfrac{2}{3}+\tfrac{2}{3}(1-a^2)^{\frac{1}{2}}\},$$
$$H_3=28a^{-7}\{\tfrac{2}{3}a^3-\tfrac{1}{4}\sin^{-1}a+\tfrac{1}{4}a(1-a^2)^{\frac{1}{2}}-\tfrac{1}{5}a^3(1-a^2)^{\frac{1}{2}}-\tfrac{1}{5}a^5\},$$
$$H_4=144a^{-9}\{8a-\tfrac{8}{3}a^3+\tfrac{1}{5}a^5-\tfrac{7}{2}\sin^{-1}a-\tfrac{8}{3}a(1-a^2)^{\frac{1}{2}}+a^3(1-a^2)^{\frac{1}{2}}\};$$

(II) $S_0=(3a^2/20)H_1$, with Δ, 8 decimals, $a=0(0.001)0.999$; (III) s, t, u, v, 6 decimals, $m=0$ to 0.995, $a=0$ to $(1-m^2)^{\frac{1}{2}}$, with interval in each case 0.02, 0.01 or 0.005; these functions are ratios of various quantities for an ogive to those for the circumscribing cylinder,

$$s=1-\frac{a^2}{3(1-m)}+\frac{ma^4}{20(1-m)^2}H_1,$$
$$t=s^{-1}\left\{1-\frac{a^2}{2(1-m)}+\frac{ma^4}{12(1-m)^2}H_2\right\};$$

further, more complicated expressions for u and v are also given; (IV) s, t, u, v, 6 decimals, $m=(1-a^2)^{\frac{1}{2}}$, $a=0(0.001)$-0.999.

Tables V and VI give functions for a frustum of a right-circular cone: (V) s, with Δ, 8 decimals, $x=0(0.001)0.999$; (VI) s, t, u, v, 6 decimals, $x=0(0.01)0.99$, where

$$s=\tfrac{1}{3}(x^3+x+1),\quad t=\tfrac{1}{4}s^{-1}(3x^2+2x+1),$$
$$u=\tfrac{1}{5}s^{-1}(x^4+x^3+x^2+x+1),$$
$$v=\tfrac{1}{10}s^{-2}\{(1+x)^4+4x^3\}.$$

Table VII gives functions for solids of revolution obtained from a fillet; they are s, t_{1A}, t_{2A}, 4 decimals, $\theta=30°(1°)120°$, where

$$s=\frac{\tan\tfrac{1}{2}\theta-\tfrac{1}{2}\theta}{\tan\tfrac{1}{2}\theta\cdot(1-\cos\theta)},$$
$$t_{1A}=1-\tfrac{1}{4}s^{-1},\ t_{2A}=t_{1A}/(1-\cos\theta).$$

Part I of the introduction gives extensive formulae connected with the functions tabulated, and gives four small tables of further functions connected with fillets. Parts II to IV describe method of computation, interpolation and the use of the tables. Part V deals with nonstandard ogives.

J. C. P. Miller (London).

Kopal, Zdeněk. Theory and tables of the associated alpha-functions. Harvard College Observatory Circular, no. 450, 25 pp. (1947).

In a series of papers [Proc. Amer. Philos. Soc. **85**, 399–431 (1942); **89**, 517–530 (1945); Astrophys. J. **94**, 159–170 (1941); **96**, 20–27 (1942); these Rev. **4**, 73; **7**, 224; **4**, 73] the author has shown how the theoretical light curves and radial velocity curves of close binary systems may be expressed in terms of "associated α-functions." These are defined for integral orders m and n in such a way that $\pi r_1^{m+n+2}\alpha_n^m$ is the integral of $x^m z^n$ taken over the area of overlap of the projections of the two discs, r_1 being the radius of the disc undergoing eclipse. Here x and y are coordinates in the plane of projection, with origin at the intersection of the line of centres with the common chord (partial eclipse) or with the diameter perpendicular to the line of centres (annular eclipse); x is measured parallel to the line of centres and $z^2=r_1^2-x^2-y^2$. The α-functions depend in general on two nondimensional variables, so that tabulation for many values of m and n presents difficulties. For the cases he requires the author has succeeded in separating the variables, so that values may be obtained as products of functions of one variable, one only not being elementary. The paper gives tables of various orders of this last function, together with mathematical theory.

The functions and tables fall into two distinct sets, according to the nature of the eclipse (partial or annular). Table I(p) contains the first basic family of functions of the type $I^m_{-1,0}$, $m=0(1)5$, appropriate for partial eclipses; these functions are the integrals $\int_0^\alpha\cos^m\theta d\theta$, and are given for $\alpha=0°(5°)180°$ to 5 decimals, with modified second differences. They are connected by the recurrence formula $mI^m_{-1,0}=(m-1)I^{m-2}_{-1,0}+\sin\alpha\cos^{m-1}\alpha$. Tables II(p) and II(a) give $I^m_{-1,1}$ for the partial and annular cases, respectively, for

$m=0(1)5$, $\alpha=0°(5°)180°$ to 5 decimals, with modified second differences, except where α is small and m even, when the functions become very large. These functions depend on the complete elliptic integrals; the first two in each case are as follows:

(p) $\pi I^0_{-1,1}=2\{2E-(1+\mu)F\}$, $3\pi I^1_{-1,1}=2\{(1+\mu)F-2\mu E\}$, $\kappa^2=\frac{1}{2}(1-\mu)$, $\alpha=2\sin^{-1}\kappa$,

(a) $\pi\kappa I^0_{-1,1}=4E$, $3\pi\kappa I^1_{-1,1}=4\{(1+\mu)F-\mu E\}$, $\kappa^2=2/(1-\mu)$, $\alpha=2\sin^{-1}\kappa$.

Expressions for the further functions tabulated are given in the paper.

Higher functions may be derived from the formulas (the same for partial or annular eclipses):

$$I^m_{\beta+2,\gamma}=I^m_{\beta,\gamma}-I^{m+2}_{\beta,\gamma},\quad I^m_{\beta,\gamma+2}=2\{I^{m+1}_{\beta,\gamma}-\mu I^m_{\beta,\gamma}\}.$$

These have been used to obtain values given in tables III to V. Each table is divided into two parts, labelled (p) for partial eclipses and (a) for annular eclipses. The functions given are $I^0_{\beta,\gamma}$; $\beta=-1$, $\gamma=2(1)7$; $\beta=1$, $\gamma=-1(1)6$; $\beta=3$, $\gamma=-1(0)3$. Values are to about 5 figures or decimals, with modified second differences always to a figure less (so that care is needed in use) except where the function is varying too rapidly. Graphs illustrating the run of the functions are also given. *J. C. P. Miller* (London).

Nelšuler, L. Ya. Remarks on tabulation. Trav. Inst. Math. Stekloff **20**, 113–116 (1947). (Russian)

Two notes are given. The first describes a proposed multiplication table giving products of four-digit numbers by two-digit numbers, and its pedestrian uses. The second note deals with tables of linear functions as used in compass correction problems. *D. H. Lehmer* (Berkeley, Calif.).

Nelšuler, L. Ya. On the tabulation of functions of three variables. Trav. Inst. Math. Stekloff **20**, 87–108 (1947). (Russian)

Nelšuler, L. Ya. On the tabulation of functions of four and many variables. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1543–1560 (1947). (Russian)

The ideas set forth in previous papers [cf. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 597–608; these Rev. **9**, 104, where further references are given] are reiterated and enlarged upon. The discussion is only general. Differential forms of the functions capable of tabulation by the various schemes are given.

D. H. Lehmer (Berkeley, Calif.).

Šatrovskiĭ, L. I. Application of the method of Nelšuler to the construction of tables for aerial gunnery. Trav. Inst. Math. Stekloff **20**, 109–112 (1947). (Russian)

The ideas of Nelšuler [cf. the preceding review] are applied to the problem of condensing a ballistic table. A reduction to $\frac{1}{3}$ the size of an ordinary table of three independent variables is claimed. *D. H. Lehmer.*

Akušskiĭ, I. Ya. Mathematical tables on punched cards. Functions of one variable. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1405–1454 (1947). (Russian)

In this address the author describes well-known methods in very great detail for the production of punch card tables. The discussion is general and lengthy. Illustrations are confined to a table of squares and four figure logarithms. There is a general discussion of interpolation from differences. *D. H. Lehmer* (Berkeley, Calif.).

Akušskiĭ, I. Ya. The operational cycles of tabulators with vertical-horizontal action. Doklady Akad. Nauk SSSR (N.S.) **59**, 1375–1378 (1948). (Russian)

The author describes in very general terms the programming of a punch card tabulator, possibly a 405, for the problem of computing the kth differences of a function. No definite examples are given, nor is there any indication of possible values of k. *D. H. Lehmer* (Berkeley, Calif.).

Cowden, Dudley J. Simplified methods of fitting certain types of growth curves. J. Amer. Statist. Assoc. **42**, 585–590 (1947).

The author's summary is as follows. A method of selected points and a graphic method is presented for fitting the modified exponential, the Gompertz, and the logistic. The procedure is illustrated for the logistic.

T. N. E. Greville (Washington, D. C.).

Michalup, Erich. Theorie und Anwendung der "oskulatorischen" Interpolationsformeln. Mitt. Verein. Schweiz. Versich.-Math. **47**, 359–407 (1947).

Certain smooth-junction formulas for interpolating equidistant data are developed and compared with previously published formulas. *T. N. E. Greville.*

Kincaid, W. M. Note on the error in interpolation of a function of two independent variables. Ann. Math. Statistics **19**, 85–88 (1948).

An expression is obtained for the remainder in interpolation of a function of two variables by an arbitrary interpolatory function satisfying the condition that it shall be equal to the given function at an arbitrary number of specified points. A more simple and elegant, and equally general, expression is obtainable by a simple adaptation of formula (41), p. 215, of J. F. Steffensen's book [Interpolation, The Williams and Wilkins Co., Baltimore, 1927].

T. N. E. Greville (Washington, D. C.).

* Please read p 855, Errata, these Math. Rev. V.

Salzer, Herbert E. Tables of coefficients for interpolating in functions of two variables. J. Math. Physics **26**, 294–305 (1948).

Tables are given of the coefficients required for interpolating to tenths of an interpolation interval for each variable by a method given in a previous note [Bull. Amer. Math. Soc. **51**, 279–280 (1945); these Rev. **7**, 85] using polynomials of total degree 2, 3 and 4 in both variables. The points used in the formula form a lattice in the pattern of a right triangle, with the desired point in the corner square. *T. N. E. Greville* (Washington, D. C.).

Adcock, Willis A. An automatic simultaneous equation computer and its use in solving secular equations. Rev. Sci. Instruments **19**, 181–187 (1948).

The method used in this paper for solving simultaneous linear equations differs from the Gauss-Seidel iteration method in that, instead of adjusting one variable to make the ith error $\epsilon_i=\sum_{j=1}^n a_{ij}x_j-b_i$ equal to zero, all the variables are simultaneously adjusted in proportion to their coefficients in the ith equation. This is accomplished by using $x_j=-\sum_{i=1}^n E_i a_{ij}$ and changing the E_i. The feed-back, which is used to make the computer automatic, is set up by the condition $dE_i/dt=K\epsilon_i$. The error of the solution is measured by $\mu=\sum_{i=1}^n \epsilon_i^2$ and when $\det(a_{ij})\neq 0$ can be shown to have a negative time derivative unless all the ϵ_i are zero. "This method of adjusting the variables places no restrictions on the type of equations which can be handled and is

thus more generally applicable than is the Gauss-Seidel method. This method can easily handle equations of the type $\sum_{j=1}^{n}(a_{ij}-b_{ij}\lambda)x_j=0$, $i=1, \cdots, n$, wherein various values are tried for λ."

The mechanization of the conditions given by the first three equations is accomplished by representing the x_i and E_i by d.c. voltages and the coefficients a_{ij} and b_i by settings of potentiometers. Each coefficient a_{ij} is needed twice, once for each of the first two equations. The two are set at the same time by mechanically ganged potentiometers. The feedback condition can be realized either by a feed-back amplifier or by a motor driven potentiometer which is controlled by an amplified error signal ϵ_i. To solve secular equations additional ganged potentiometers are needed for the λ terms. A small 4-equation model was built to solve secular equations. On this model the λ potentiometers are used to set the coefficient potentiometers, a process which requires around ten minutes for the whole set. The reported solutions obtained on this model indicate a reliability of better than 1% and include one with multiple latent roots. It is stated that to scan the region $\lambda=0.6$ to $\lambda=0.25$ requires about three minutes. *R. W. Hamming.*

Stearn, Joseph L., and Braaten, Norman F. A method of simultaneous solution of a system of observation and condition equations. Trans. Amer. Geophys. Union **29**, 157–162 (1948).

The authors deduce a new method for obtaining a solution of a system of observation equations which is constrained by a number of conditions. Let A, B be r by M and k by r matrices, respectively, and x, b, v, w column vectors of dimensions $M=2m$, r, r, k. Further, let the r observation equations be $Ax+b=v$. Then the minimum condition for v^*v subject to the k condition equations $Bv+w=0$ yields the system of normal equations $A^*Ax+A^*B^*\lambda+A^*b=0$, $BAx+Bb+w=0$, where λ is the column vector of dimension k formed by the Lagrangian multipliers λ_i. The method is well adapted to adjustment problems in area triangulation. *E. Bodewig* (The Hague).

von Neumann, John, and Goldstine, H. H. Numerical inverting of matrices of high order. Bull. Amer. Math. Soc. **53**, 1021–1099 (1947).

The authors "wish to determine the precision and stability of a (long) computation with respect to the round off errors." They take as an example the problem of inverting high order matrices ($n\sim100$) by the elimination method. This is the most appropriate method for their aim, first because the inversion of such matrices plays an important role in the introduction of new procedures, second because it shows the problem of accumulation of round errors in "Reinkultur," as it is an elementary one, having no difficulties of transcendentality, like differential equations. As for high-speed calculating machines, otherwise than for normal machines, a continuous inspection and supervision of the calculation is undesirable or impossible, the authors restrict the analysis to definite matrices since only for them can the question of precision be put on a rigorous basis. Then an arbitrary matrix M must first be transformed into a definite one $M'M$. The case of a definite matrix A is treated by an elaborate rigorous analysis. Its essential tricks are as follows. First, A can be decomposed into $A=B'DB$, where B is triangular and D is diagonal. Second, even when the resulting approximations of B and D to their "true" values are poor, the decomposition itself is still good within an error of $0.42n^2\times10^{-s}$ in terms of the bound of the matrix $(A-B'DB)$, where s denotes the fixed number of digits used in the calculation. This estimate is only possible for a definite A. Finally the authors obtain an approximant 2^sW to A^{-1} and show that the approximation is good within an error at most $14(\lambda/\mu)n^2\cdot10^{-s}$, where λ and μ are the largest and smallest proper values of A. The algorithm itself is such that either the approximate inverse is found or it is discovered that $\mu<n^2\cdot10^{-s+1}$, i.e., that A is approximately singular. For a general orientation about the conditions that prevail for nth order matrices, the authors use statistical estimates of λ, μ and put $\lambda\leqq100n$, $\mu\geqq1/(100n)$. Thus an approximate inverse will usually be found if $n<0.15\times10^{s/4}$. By this result the usual opinions as to the precision required when inverting matrices of high order are refuted. *E. Bodewig* (The Hague).

Magnier, André. Sur le calcul numérique des matrices. C. R. Acad. Sci. Paris **226**, 464–465 (1948).

Let the n distinct principal directions of a symmetric matrix A be known approximately. From them n orthogonal unit vectors are derived by using Schmidt's process of orthogonalization. If K is the matrix of the transformation of coordinates belonging to them, the matrix $B=K^*AK$ has a predominant diagonal. In the new system the principal directions are given in the first approximation by the matrix $E+M$, where $m_{ik}=b_{ik}/(b_{kk}-b_{ii})$. Iterating this process a sequence is obtained which will converge rapidly to the principal directions of A. The author prefers this iteration process to the methods proposed, for example, by Kincaid [see Quart. Appl. Math. **5**, 320–345 (1947); these Rev. **9**, 210]. *E. Bodewig* (The Hague).

Dimsdale, Bernard. On Bernoulli's method for solving algebraic equations. Quart. Appl. Math. **6**, 77–81 (1948).

Since in Bernoulli's method the quotient s_{n+1}/s_n of the powers of the roots of $p(z)$ does not converge if there are several roots of the same largest modulus, the author generalizes this method and determines a factor $P(z)$ of $p(z)$ the roots of which are all simple and are the unequal roots $z_1, \cdots, z_M$ of the largest moduli, where $|z_M|>|z_{M+1}|$. For the α_i of $P(z)=z^M+\alpha_1z^{M-1}+\cdots+\alpha_M$ are the limits of certain α_{in}: $\alpha_i=\alpha_{in}+O(z_{M+1}/z_M)^n$, where the α_{in} are the solutions of the linear system $s_p+\alpha_{1n}s_{p-1}+\cdots+\alpha_{Mn}s_{p-M}=0$, $p=n, n+1, \cdots, n+M-1$. *E. Bodewig* (The Hague).

Goodstein, R. L., and Broadbent, T. A. A. The convergence of iterative processes. J. London Math. Soc. **22** (1947), 168–171 (1948).

This is a contribution to the solution of an equation $\psi(x)=x$ by iterative processes. Known results are made more precise in the following form. Let $\phi(x)=x-\psi(x)$. For the field of complex numbers: if $c_0=c$ (arbitrary) and $c_{n+1}=c_n-\phi(c_n)$, then c_n converges and its limit is a root of $\phi(x)=0$ if (1) $\phi(x)$ is analytic in the circle Γ_c: $|x-c|\leqq|\phi(c)|/(1-\kappa)$, $0<\kappa<1$, (2) there is a constant δ, such that $0<\delta<\kappa<1$, $|1-\phi'(c)|<\delta$, and

$$|\phi''(x)|<(\kappa-\delta)(1-\kappa)/|\phi(c)|$$

in the circle Γ_c. For the field of real numbers: replace (1) and (2) by (1′) $\phi''(x)$ exists in the interval $|x-c|\leqq|\phi(c)|/(1-\kappa)$, $0<\kappa<1$, (2′) there is a constant δ, such that $0<\delta<\kappa<1$, $|1-\phi'(c)|<\delta$ and $|\phi''(x)|<(\kappa-\delta)(1-\kappa)/|\phi(c)|$ in the interval $|x-c|\leqq|\phi(c)|/(1-\kappa)$, $0<\kappa<1$. *A. J. Kempner.*

Hruška, Václav. A contribution to the solution of systems of equations by iteration. Rozpravy II. Třídy České Akad. **53**, no. 6, 38 pp. (1943). (Czech)

Hruška, Václav. A second contribution to the solution of systems of equations by iteration. Rozpravy II. Třídy České Akad. **53**, no. 17, 68 pp. (1943). (Czech)

Hruška, Václav. A third contribution to the solution of systems of equations by iteration. Rozpravy II. Třídy České Akad. **53**, no. 32, 25 pp. (1943). (Czech)

For a German summary see Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 239–304, 399–422 (1943); these Rev. **8**, 605, 606.

Mihlin, S. G. On the convergence of the method of least squares. Doklady Akad. Nauk SSSR (N.S.) **59**, 1245–1247 (1948). (Russian)

Consider

$$L_i(x_1, \cdots, x_m)=0, \qquad i=1, \cdots, l; \tag{1}$$

$$G_i(x_1, \cdots, x_m)=b_i, \qquad i=1, \cdots, n, \tag{2}$$

in a separable Hilbert space H, where L_i and G_i are linear operators. An approximate solution of (1) and (2) is built up as follows:

$$x_i^{(p)}=\sum_{k=1}^{p} a_k^{(p)} \chi_i^k, \qquad i=1, \cdots, m,$$

where the sequence $\{\chi_1^k, \cdots, \chi_m^k\}$ is a solution of (1) and the $a_k^{(p)}$'s are determined from the condition

$$\sum_{i=1}^{n} \|G_i(x_1^{(p)}, \cdots, x_m^{(p)})-b_i\|^2=\text{minimum}. \tag{3}$$

The sequence $\{\chi_1^k, \cdots, \chi_m^k\}$ is called absolutely-complete if

$$\sum_{i=1}^{n} \|G_i(x_1, \cdots, x_m)-G_i(\chi_1^{(p)}, \cdots, \chi_m^{(p)}\|^2<\epsilon^2,$$

where $\epsilon>0$, p is a positive integer and $\alpha_1, \cdots, \alpha_p$ are constants determined by

$$\chi_i^{(p)}=\sum_{k=1}^{p} \alpha_k \chi_i^k, \qquad i=1, \cdots, m.$$

Lemma 1. The method of least squares converges provided: (A) the sequence $\{\chi_1^k, \cdots, \chi_m^k\}$ is absolutely-complete; (B) the system (1) and (2) possesses a solution and this is unique; (C) if $x_1, \cdots, x_m$ satisfy (1), then

$$\sum_{i=1}^{m} \|x_i\|^2 \leqq K^2 \sum_{i=1}^{n} \|G_i(x_1, \cdots, x_m)\|^2,$$

K being a constant; (D) condition (3) uniquely determines the constants $a_k^{(p)}$. Lemma 2. For the case $m=n$, condition (D) is a consequence of the other conditions. Lemma 2 implies the convergence of the least squares method for a linear system $Ax=b$, the operators A and A^{-1} being bounded. By a suitable modification the method can still be made to converge even when A^{-1} does not exist, but A can be "regularized"; that is, there exists a bounded linear operator M, such that $MA=E+T$, where E is an identical and T a completely continuous operator.

As applied to problems of mathematical physics, (1) corresponds to the differential equation, while (2) expresses the boundary conditions. The method of least squares is applicable also to the solution of integral equations of the Fredholm type as well as of singular ones. The method has been used by the author for solving the problems of Dirichlet for vibratory motion and of Neumann in the plane as well as in three-dimensional space. *M. Daniloff.*

Vernotte, Pierre. A propos du calcul pratique de la limite d'une variable. C. R. Acad. Sci. Paris **225**, 1130–1132 (1947).

Thoughts on the technique of extrapolation to infinity, particularly in connection with divergent series.

J. W. Tukey (Princeton, N. J.).

Ciurileanu, D. I. Sur un nouvel appareil pour la mesure des surfaces: "le parcellmètre." Acad. Roum. Bull. Sect. Sci. **25**, 327–332 (1943).

The parcel-meter consists of a pivoted cursor attached to a rider on a graduated fixed scale, similar in construction to a drafting machine. It is used to mechanize the graphical reduction of a polygon to an equivalent triangle in order to measure the area. Since it makes use only of the vertices of the polygon and it is not necessary to trace the perimeter, it is claimed to be more accurate and faster than a planimeter.

M. Goldberg (Washington, D. C.).

Merli, Luigi. Su una formula di quadratura. Boll. Un. Mat. Ital. (3) **2**, 132–134 (1947).

Si $f(x)$ est une fonction continue dans $(-1, 1)$, on a:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^{(n)}) \int_{-1}^{1} \{l_k^{(n)}(x)\}^2 dx=\int_{-1}^{1} f(x)dx,$$

où les $x_k^{(n)}$ sont les zéros du polynome $T_n(x)=\cos(n \cos^{-1} x)$ et $l_k^{(n)}$ le polynome qui prend la valeur 1 pour $x_k^{(n)}$ et s'annule pour tous les autres zéros de T_n. *J. Favard* (Paris).

Segal, B. I. On a problem of heat conduction. Trav. Inst. Math. Stekloff **20**, 65–76 (1947). (Russian)

A straight wire of circular cross section lies in an infinite homogeneous medium. The ends of the wire are kept permanently at zero temperature. The initial temperature of the medium is also zero. Neglecting the variation of the temperature within the cross section of the wire and using a result of S. Goldstein [Proc. London Math. Soc. (2) **34**, 51–88 (1932)] and the theorem of Duhamel the author expresses the temperature in the medium as a double integral whose integrand contains Bessel functions of zero order and the time derivative of the temperature of the wire. The calculation of the latter is reduced to the solution of the equation in $u(x, t)$:

$$au_{zz}-u_t=b\int_0^t K(t-z)u_z(z, t)dz-1,$$

where $u(\pm c, t)=u(x, 0)=0$; a, b, c are constants and K is essentially the Laplace transform of $\{t[J_0(\sqrt{t})]^2+t[Y_0(\sqrt{t})]^2\}^{-1}$. The author gives an approximate solution of the problem by means of finite differences. *I. Opatowski.*

Ditkin, V. A. Solution of a problem of heat conduction by the method of operational calculus. Trav. Inst. Math. Stekloff **20**, 77–86 (1947). (Russian)

The equation of the paper reviewed above is solved by means of a Laplace transform taken with respect to t. Series expansions and definite integrals for the calculation of $u(x, t)$ are given. These calculations are simplified by a numerical table of $xH_1^{(1)}(x)/H_0^{(1)}(x)$ for $x=0.02(0.02)1$ and $1.1(0.1)9.9$, where the H's are Hankel functions.

I. Opatowski (Ann Arbor, Mich.).

Eichler, M. Auflösung der Integralgleichung von Possio für den harmonisch schwingenden Tragflügel im kompressiblen Medium durch Zurückführung auf ein lineares Gleichungssystem. Jahrbuch 1942 der Deutschen Luftfahrtforschung, I169–I172 (1942).

Possio [Aerotecnica 18, 441–458 (1938)] has formulated the problem of compressible flow past an oscillating thin wing of infinite span as a linear integral equation. In dimensionless form, this equation becomes $f(x)=\int_{-1}^{1}K(x-\xi)\varphi(\xi)d\xi$, where $\varphi(\xi)$ is the unknown "dipole distribution" to which the solution of the flow problem is reduced, $f(x)$ is essentially the profile of the cross-wind velocity amplitude of the wing and $K(x-\xi)$ is a singular kernel, known as an explicit function of $x-\xi$ and of the constants of the problem, viz., the Mach number, oscillation frequency and chord length. This paper replaces the integral equation formally by an infinite system of linear equations in which the coefficients are known in terms of the a_n of the expansion $f(x)\sin z=\sum_{n=1}^{\infty}a_n\sin nz$, and the unknowns are the coefficients b_n in $\varphi(x)\sin z=\sum_{n=0}^{\infty}b_n\cos nz$, where $x=\cos z$, $0\leqq z\leqq\pi$. A method is indicated by which approximate solutions can be obtained from a finite subsystem of the entire system of equations. *D. Gilbarg.*

Akušskiĭ, I. Ya., and Ditkin, V. A. On the numerical solution of the equation of circulation of an oscillating wing. Trav. Inst. Math. Stekloff 20, 7–38 (1947). (Russian)

An integral equation for $\Gamma(y)$ of the form

$$\Gamma(y)=f(y)+\int_{-s/\lambda}^{s/\lambda}K(y-\eta)d\Gamma(\eta),$$

$\Gamma(-s)=\Gamma(s)=0$; $\Gamma(-y)=\Gamma(y)$ $(-s\leqq y\leqq s)$; $\lambda>1$, is reduced to infinitely many linear equations with an infinite number of unknowns, by expansion of $f(y)$ and $\Gamma(y)$ in Fourier series. The coefficients of these equations contain integrals involving products of Bessel functions. The authors give a detailed description of the numerical calculation of a number of these integrals by means of Simpson's rule with a punched-card machine (tabulator). *W. H. Muller.*

Koehler, J. S. An electronic differential analyzer. J. Appl. Phys. 19, 148–155 (1948).

A method for electronic analogue computation is described, based on the use of the charge on the condenser of a series resonant circuit to represent the behavior of the dependent variable of a second-order differential equation. For the representation of higher order equations additional circuit elements must be used. The required variation of capacitance and resistance is obtained by means of special feed-back amplifiers which have bias voltages controlled by variable-voltage generators. The latter device makes use of a cathode-ray tube, opaque mask, photocell and amplifier to produce a predetermined voltage variation. Only parts of the machine have been completed and overall performance data are hence not available. *S. H. Caldwell.*

Shaw, F. S. The approximate numerical solution of the non-homogeneous linear Fredholm integral equation by relaxation methods. Quart. Appl. Math. 6, 69–76 (1948).

Relaxation methods are applied to the problem of finding an approximate numerical solution to the non-homogeneous linear Fredholm integral equation. As an illustration of the technique the deflection of a simply supported single span beam subjected to both normal and end loads is found.

Author's summary.

Smith, F. C. The force of mortality function. Amer. Math. Monthly 55, 277–284 (1948).

MECHANICS

Dobrovol'skiĭ, V. V. On the motion of the center of gravity of hinged four-linkages. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] 1941, no. 4, 107–108 (1941). (Russian)

It is shown geometrically that the locus of the combined center of gravity of the three moving links of a four-bar linkage is homothetic to the curve traced by a particular point on the connecting-rod. *M. Goldberg.*

Artobolevskiĭ, I. I., and Abramov, B. M. On a form of the equations of motion of a machine. Doklady Akad. Nauk SSSR (N.S.) 59, 1261–1264 (1948). (Russian)

The angular velocity ω of the driving link of a rotating or vibrating mechanism varies with the following quantities: the moment of inertia J, and the difference ΔM between the driving moment and the resisting moment, each of which varies with the position angle ϕ. The motion is solved by a step-by-step integration process in which constant "fictitious" values (designated by the zero subscripts) are used over intervals in accordance with the reduced equation $\Delta M_0=J_0 d\omega_0/dt$. The method is illustrated by a practical graphical example. *M. Goldberg* (Washington, D. C.).

Rajagopal, C. T. Concerning reciprocal screws. Math. Student 14 (1946), 75–76 (1948).

Wunderlich, W. Höhere Radlinien. Österreich. Ing.-Arch. 1, 277–296 (1947).

Es liegt ein erster Kreis vor, der fest ist und auf dem ein zweiter rollt, auf diesem rollt dann ein dritter, auf diesem wiederum ein vierter, u.s.w. Alle Geschwindigkeiten sind beliebig, aber konstant. Ein mit dem n-ten Kreis starr verbundener Punkt beschreibt eine Bahn, die Radlinie n-ter Stufe heissen soll. Die Radlinien zweiter Stufe sind die Zykloiden. Es handelt sich offenbar um die Bahnkurven der Epizykeltheorie des Ptolemäusschen Planetensystems. Der Verf. gibt die Gleichungen der Kurven in der komplexen Ebene. Jede Radlinie n-ter Stufe kann auf $n!$ Arten durch ein zwangläufig bewegliches n-gliedriges Gelenkpolygon erzeugt werden. Die Polbahnen einer Radlinie n-ter Stufe sind Radlinien $(n-1)$-ter Stufe und jede Radlinie n-ter Stufe kann auf n Arten durch das Abrollen zweier Radlinien $(n-1)$-ter Stufe erzeugt werden. Es werden die algebraischen Radlinien betrachtet, die Hüllbahnen von Geraden bei der Bewegung, die Evoluten und Evolventen, die isoptischen Kurven und die Fusspunktskurven.

O. Bottema (Delft).

Prosciutto, Aristide. Problemi geometrici riguardanti la costruzione degli ingranaggi per assi sghembi. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) 8, 133–140 (1941).

***Chazy, Jean. Cours de Mécanique Rationnelle. Tome II. Dynamique des Systèmes Matériels.** 3d ed. Gauthier-Villars, Paris, 1948. vi+511 pp. 1100 francs.

Based on the author's first volume [published 1947; these Rev. 8, 413] which covered the dynamics of a particle, this

volume discusses systems of particles, rigid bodies, strings, hydrostatics, hydrodynamics and gravitational potential. The treatment is clear, concise and exact. Free use is made of vector notation. There are some illustrative problems worked out in the text, but relatively few for many of the topics. For instance, besides a few cases discussed in the chapter on small oscillations, the only specific illustration of Lagrange's equations given is a brief discussion of the symmetrical top. No exercises to be worked out are included. Although somewhat shaped by its limited objective, this volume should prove valuable to any reader who desires a brief theoretical discussion of advanced dynamics.

P. Franklin (Cambridge, Mass.).

Synge, J. L. Apsidal angles for symmetrical dynamical systems with two degrees of freedom. Bull. Amer. Math. Soc. **54**, 111–118 (1948).

The azimuthal increment α of the bob of a spherical pendulum as it passes from its lowest point to its highest is well known to satisfy the inequalities $\frac{1}{2}\pi<\alpha<\pi$. Similar inequalities are obtained by the author for the general conservative dynamical system with two degrees of freedom and one ignorable angular coordinate. The bounds obtained are dependent on the particular system. It is shown, in fact, that for the whole class of such systems there is neither an upper bound nor a lower positive bound for α, the so-called apsidal angle. There is also obtained a formula for α in the form of a surface integral of the Gaussian curvature for a suitable metric over a certain triangle. This formula is obtained from a classical result on the excess over π of the sum of the angles of a geodesic triangle. *D. C. Lewis.*

Ghermănescu, Michel. Sur le mouvement tautochrone plan. Acad. Roum. Bull. Sect. Sci. **23**, 411–413 (1942).

The author derives relations between a law of force toward a center assumed to be of the form $f(\theta)$, and a family of plane curves resulting in tautochronous motion.

P. Franklin (Cambridge, Mass.).

Ghermănescu, M. Mouvements d'un solide autour d'un point fixe. Acad. Roum. Bull. Sect. Sci. **25**, 1–3 (1943).

Bădesco, Radu. Sur une extension du mouvement tautochrone plan. Acad. Roum. Bull. Sect. Sci. **24**, 89–93 (1943).

The extension consists in replacing s in the tautochronous equation by $F(s)$, such that $F(0)=0$ and $F'(0)>0$, and studying the motion near the origin. *P. Franklin.*

Vâlcovici, Victor. Sur certains mouvements ayant des trajectoires planes. Acad. Roum. Bull. Sect. Sci. **23**, 414–419 (1942).

Relations on k_i and k', each functions of the components of the position and velocity vectors and the time, are determined so that the trajectories of the system $r=\sum_{i=1}^{n}k_i(r-r_i)+k'v$ due to n central forces will all be plane curves. *P. Franklin* (Cambridge, Mass.).

Cioranescu, Nicolas. Le mouvement d'un point matériel dans un milieu résistant sous l'action d'une force centrale. Acad. Roum. Bull. Sect. Sci. **23**, 127–131 (1942).

Sălceanu, Constantin, et Borneas, Marius. Sur le couplage de deux pendules de résonance. C. R. Acad. Sci. Paris **226**, 874–876 (1948).

Hostinský, Bohuslav. The acoustic spectrum of a string. Rozpravy II. Třídy České Akad. **53**, no. 3, 29 pp. (1943). (Czech)

Hostinský, Bohuslav. On the distribution of energy in acoustic spectra. Rozpravy II. Třídy České Akad. **53**, no. 31, 17 pp. (1943). (Czech)

German summaries appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 23–32, 393–398 (1943); these Rev. **9**, 110.

Sneddon, Ian N. Daniel Bernoulli's mechanical problem. Philos. Mag. (7) **39**, 229–235 (1948).

The uses of the finite Hankel transformation

$$\int_0^\infty xf(x)J_0(\lambda x)dx=\bar{f}(\lambda_i),$$

where λ_i are the positive zeros of the function $J_0(\lambda a)$, are illustrated. Specific boundary value problems in the free and forced transverse displacements $y(x, t)$ of a heavy chain, suspended from one end, are solved. The corresponding problem in the transform $\bar{y}(\lambda_i, t)$ is a simple initial value problem in ordinary differential equations. The infinite Hankel transform is used to solve similar problems for semi-infinite chains. *R. V. Churchill* (Ann Arbor, Mich.).

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. On the total vorticity of motion of a continuous medium. Physical Rev. (2) **73**, 510–512 (1948).

The total vorticity of a moving continuous medium is the volume integral of the vorticity vector. A general kinematical formula for the rate of change of total vorticity is obtained, from which the following theorems are deduced. (i) The total vorticity of a motion filling all space is constant in time, provided that the motion vanishes at infinity to a sufficiently high order; (ii) the rate of change of total vorticity in an arbitrary volume is independent of the state of motion at all interior points of the volume. The mechanism of diffusion of total vorticity is discussed in the case of a viscous fluid. *A. J. McConnell* (Dublin).

Truesdell, C. On the transfer of energy in continuous media. Physical Rev. (2) **73**, 513–515 (1948).

A formula for the rate of change of energy of a moving compressible viscous fluid is obtained which is a generalisation of the theorem of Bobyleff and Forsyth [Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 580–581]. *A. J. McConnell* (Dublin).

Eckart, Carl. The theory of the anelastic fluid. Rev. Modern Physics **20**, 232–235 (1948).

In an earlier paper [Physical Rev. (2) **73**, 373–382 (1948); these Rev. **9**, 394] the author developed a general theory of anelastic substances. In the present paper this theory is simplified for the case of an anelastic fluid. It is shown that slow steady flows and the diffusion of vorticity are governed by the same equations as in the Navier-Stokes theory of viscous fluids. The propagation and absorption of longitudinal isentropic waves, however, exhibits hysteresis in the response of the density to the pressure. This hysteresis tends to vanish for very low and very high frequencies, and the compressibility in these two limiting cases tends towards different real values. For intermediate frequencies

the compressibility is a complex function of the frequency. The two zeros of this function determine two relaxation times, one of which is associated with the hysteresis, the other with viscosity. *W. Prager* (Providence, R. I.).

García, Godofredo. Exact equations and exact solutions of the motion and of the tensions in viscous fluids. Actas Acad. Ci. Lima **10**, 117–170 (1947). (Spanish)

The author states that all known exact solutions of the equation of motion of a viscous fluid will have to be revised in the light of the equation of this paper. No derivation of the equation is given and as it contains at least two undefined symbols it is difficult to judge the author's intention. Cartesian scalar components of the equation are given in extenso, but no solution, exact or otherwise, is offered.

L. M. Milne-Thomson (Greenwich).

Fainzil'ber, A. M. Questions of thermal modelling. Doklady Akad. Nauk SSSR (N.S.) **59**, 697–700 (1948). (Russian)

The present paper deals with the problem of the thermal phenomena occurring in an aggregate of viscous fluids in motion. Starting with the equation of the conservation of energy and the hydrodynamic equation, there results the differential equation

$$(*)\qquad \tau_1\frac{d^2T_1}{du_1^2}+(1-P_r)\frac{d\tau_1}{du_1}\frac{dT_1}{du_1}+(\mu-1)P_r\bar{B}^2\tau_1=0,$$

in which the symbols have the following meanings: u and v are the velocity components in the x and y directions, respectively, and T the temperature, while $\bar{u}$ and $\bar{v}$ are the velocity components of the adiabatic flow and $\bar{T}$ the corresponding temperature, $u_1=u/\bar{u}$, $T_1=T/\bar{T}$, $\tau_1=\tau\mu^{-1}$; the symbols ρ, μ and τ have the usual meanings of density, kinematic viscosity and intensity of viscous shear, respectively; furthermore, P_r is Prandtl's number and $\bar{B}$ the Mach (Barstow) number. Putting $\tau_1=A_0(1-u_1^3)$ so as to satisfy the boundary conditions, (*) reduces to:

$$\frac{d^2T_1}{du_1^2}-\frac{3bu_1^2}{1-u_1^3}\frac{dT_1}{du_1}+a=0$$

($b=1-P_r$; $a=(\mu-1)P_r\bar{B}^2$), a solution of which is given in the form of a series for T_1 in powers of u_1. Tables of T_1 are set up for water vapor and air and conclusions drawn concerning the temperature distribution obtaining in a mixture of air and water vapor in motion.

In case $\bar{B}<\frac{1}{3}$, the motion of the fluids can be taken as being isothermal. Introducing the total energy as the dependent variable $i=\frac{1}{2}u^2+\int\rho^{-1}d\rho$, and $X=\mu\int\rho dx$, $Y=\psi(x,y)$ as the independent variables ($\psi(x,y)$ being the stream function of Stokes) the general equation becomes

$$(**)\qquad \partial Q/\partial X=\{2(Q-Q_0)\}^{\frac{1}{2}}\partial^2Q/\partial Y^2.$$

The energy Q is measured from its value corresponding to an adiabatic flow taken as zero. The boundary conditions applicable to (**) are $Q=Q_0$ at $Y=0$ and $Q=0$ at $Y=\infty$. Equation (**) is general in the sense that it holds for any fluid, the various viscous continua differing only in the form of the function X. *M. Daniloff* (Cambridge, Mass.).

Oswatitsch, Kl. Über die Charakteristikenverfahren der Hydrodynamik. I, II. Z. Angew. Math. Mech. **25/27**, 195–208, 264–270 (1947). (German. Russian summary)

A general formulation of the characteristic theory in fluid dynamics with emphasis on graphical and numerical methods. The paper presents the author's own method of calculation for the case that the characteristics in the plane of the dependent variables (e.g., the hodograph plane) are not fixed, as, for example, in axially symmetric flows. His method is to reduce such a case to the well-known simpler situation in which the characteristics are fixed in the plane of the dependent variables; thus if two neighboring points P, Q in the flow plane and the corresponding values of the dependent variables are given, the flow data at the intersection of the characteristics through P, Q are computed by applying the simpler method of fixed characteristics to suitably corrected values of the given data. The author discusses the relative difficulty and accuracy of several alternative methods of computation. *D. Gilbarg.*

Vazsonyi, Andrew. A new derivation of the method of characteristics for axially symmetrical supersonic flow. Quart. Appl. Math. **5**, 499–503 (1948).

The author obtains a simple derivation of the following equation expressing the (approximate) change in magnitude and direction of the velocity vector along a characteristic in the meridian plane of an axially symmetric supersonic flow:

$$\Delta\theta=\frac{\Delta V}{V}\cot\alpha+\left(\frac{2\omega\cos\alpha}{V}-\frac{\sin\theta\sin\alpha}{r}\right)\Delta\sigma;$$

here $\Delta\sigma$ is the distance between two points A, B in the flow plane at which the flow data are known, $\Delta\theta$, ΔV are the change in direction and magnitude, respectively, of the velocity vector along a characteristic extending from one of these points to the next vertex in the characteristic net triangle determined by A and B, α is the Mach angle at this point, ω is the rotation of the flow and r is the cylindrical coordinate. Alternative derivations of the method of characteristics for axially symmetric flows have been given by others [cf., for example, the preceding review].

D. Gilbarg (Bloomington, Ind.).

Cronvich, Lester L. A numerical-graphical method of characteristics for axially symmetric isentropic flow. J. Aeronaut. Sci. **15**, 155–162 (1948).

***MacDonald, J. K. L. On some problems involving linearization of aero-thermodynamical equations.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 241–251. Interscience Publishers, Inc., New York, 1948. $5.50.

The equations for some idealized one-dimensionalized flows involving heat release are studied by the method of linearization. Lagrange coordinates are used because they have the apparent advantage of requiring relatively few terms to be dropped in order to effect linearization. An explicit problem is discussed and the results of linearization are compared with those obtained by numerical solutions (method of finite differences) of the corresponding nonlinear equations. *P. A. Lagerstrom* (Pasadena, Calif.).

Rudnev, Yu. V. On certain motions of a gas with variable entropy and total energy. Doklady Akad. Nauk SSSR (N.S.) **59**, 869–870 (1948). (Russian)

It is shown that the modified stream function associated with a certain restricted family of rotational flows obeys the same differential equation in the hodograph plane as does the (unmodified) stream function conventionally associated

with isentropic flows. The author concludes that for each isentropic flow there exist rotational flows having the same streamlines. No specific examples are discussed.

G. F. Carrier (Providence, R. I.).

Neményi, P., and Prim, R. Some properties of rotational flow of a perfect gas. Proc. Nat. Acad. Sci. U. S. A. **34**, 119–124 (1948).

The authors study the flow patterns satisfying the differential equations of rotational flow of a perfect gas and having throughout the field certain prescribed geometric, kinematic or physical properties. The patterns found are composed either of parallel straight lines, concentric circles or logarithmic spirals. *H. S. Tsien* (Cambridge, Mass.).

Falkovich, S. V. Plane motion of gas at hypersonic velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 459–464 (1947). (Russian. English summary)

The author discusses the equations of a two-dimensional supersonic potential gas flow using as independent variables the inclination θ of the velocity vector and the speed-variable $z=(M^2-1)^{-\frac{1}{2}}$, where M is the Mach number. The Legendre potential $\Phi=xu+yv-\varphi$ (where $\varphi(x, y)$ is the velocity potential, $u=\varphi_x$, $v=\varphi_y$) satisfies an equation of the form (1) $\Phi_{zz}+A(z)\Phi_{\theta\theta}+B(z)\Phi_\theta=0$, where A and B are rational functions depending upon the adiabatic exponent γ. In characteristic variables μ, λ equation (1) becomes (2) $\Phi_{\lambda\mu}-L(\mu-\lambda)[\Phi_\mu-\Phi_\lambda]$, where L is a certain function. The author shows that for large values of M ($M>4$) this function may be replaced by $L=2/(\mu-\lambda)$. With this coefficient equation (2) possesses an explicit general solution depending upon two arbitrary functions. In a similar way the equation for the stream-function ψ may be reduced to an equation of the form (2), and for large Mach numbers L may be replaced by $3/(\lambda-\mu)$. The resulting equation is again integrable in closed form.

The author also notes that for Mach numbers close to one (i.e., for large values of z) the coefficients of (1) are closely approximated by $A=-(h^2-1)^2h^{-4}z^{-2}$, $B=3/z$, and that for large Mach numbers these coefficients may be replaced by $A=-(h^2-1)^2$, $B=(h^2-2)/z$. (Here $h^2=(\gamma+1)/(\gamma-1)\approx 6$ for $\gamma=1.4$). From this fact he derives in a few lines similarity laws for transonic and hypersonic flows.

L. Bers (Syracuse, N. Y.).

Chou, P. Y. The laminar mixing motion of two incompressible gases. Chinese J. Phys. **7**, 96–101 (1947). (English. Chinese summary)

The author investigates the differential equations for the motion of a mixture of two gases with the restrictions that thermal diffusion can be neglected and that the sum of numbers of molecules of both kind per unit volume is constant. The second restriction is called by the author the condition of incompressibility. Under these restrictions, the temperature does not enter into the calculation and the unknowns of the problem are the three components of the mean mass velocity of the mixture, the pressure and the number concentration of one component of the mixture. These five unknowns are determined by five equations, three dynamic equations, the continuity equation and the diffusion equation. [The author's statement that there are six unknowns and the need for the adiabatic equation of state for gases are difficult to understand.] The author then shows that, for mixing in plane jets, the prevailing similarity law is the same as that for one gas. *H. S. Tsien.*

Kovasznay, L. I. G. Laminar flow behind two-dimensional grid. Proc. Cambridge Philos. Soc. **44**, 58–62 (1948).

Let the grid have spacing M between its (dimensionless) elements and be situated along the y-axis, and let $R=MU/\nu$ be the Reynolds number of the flow, where ν is the kinematic viscosity, and U is the velocity of the uniform flow in the x direction at a large distance from the grid. This paper shows that the stream function,

$$\psi=Uy-(UM/2\pi)e^{\lambda x/M}\sin(2\pi y/M),\quad \lambda=\tfrac{1}{2}R-(\tfrac{1}{4}R^2+4\pi^2)^{\frac{1}{2}}$$

is an exact solution of the Navier-Stokes equations, and corresponds to a flow pattern which can be considered to represent a flow behind the grid. This flow has a pair of bound vortices behind each grid element, and becomes uniform as $x\to\infty$. *D. Gilbarg* (Bloomington, Ind.).

Tollmien, W. Asymptotische Integration der Störungsdifferentialgleichung ebener laminarer Strömungen bei hohen Reynoldsschen Zahlen. Z. Angew. Math. Mech. **25/27**, 33–50, 70–83 (1947). (German. Russian summary)

The author gives the bounds of the errors committed in the usual approximate methods of solving Sommerfeld's equation for hydrodynamic stability,

$$(1)\quad \varphi^{iv}-2\alpha^2\varphi''+\alpha^4\varphi+i\alpha R[(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi]=0,$$

where φ is the unknown amplitude function, U is the (given) velocity distribution and α, c, R are parameters representing, respectively, the wave number, the complex phase velocity and the Reynolds number of the main flow. It is first shown that the equation may be approximated by

$$(2)\quad \varphi^{iv}+i\alpha R[(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi]=0,$$

which is obtained from (1) by neglecting $2\alpha^2 i/\alpha R$ against $U-c$ and $\alpha^4 i/\alpha R$ against U'' for large values of αR. The usual approximate solutions are then compared with the exact solutions of (2) and explicit formulae for the bounds of the errors for φ and its derivatives are obtained. It is shown that they are negligible for large values of αR. In the concluding section, it is demonstrated why the theory of stability of parallel flows can be applied to flows of the boundary layer type. *C. C. Lin* (Cambridge, Mass.).

Sears, W. R. The boundary layer of yawed cylinders. J. Aeronaut. Sci. **15**, 49–52 (1948).

The author considers boundary layer flow on an infinite yawed cylinder. Let z denote the coordinate parallel to the axis of the cylinder; x and y are curvilinear coordinates, x taken along the surface of a cylinder, y normal to the surface. The boundary layer equations are shown to become

$$(1)\quad \partial u/\partial t+u\partial u/\partial x+w\partial u/\partial z=-\rho^{-1}\partial p/\partial x+\nu\partial^2 u/\partial z^2,$$

$$(2)\quad \partial v/\partial t+u\partial v/\partial x+w\partial v/\partial z=\nu\partial^2 v/\partial z^2,$$

$$\partial p/\partial z=0.$$

Hence the problem can be divided into the solution of a nonlinear boundary layer equation (1) in u and w and the subsequent solution of the linear equation (2) for v. The importance of this result with regard to separation, etc., is discussed. Explicit solutions are obtained for the yawed flat plate and a class of cylinders characterized by a potential velocity distribution, $U(x)=a_1x+a_3x^3$.

H. W. Liepmann (Pasadena, Calif.).

Carrier, G. F., and Lin, C. C. On the nature of the boundary layer near the leading edge of a flat plate. Quart. Appl. Math. 6, 63–68 (1948).

The Blasius solution for boundary layer flow along an infinitely thin plate is valid only for large distances from the leading edge of the plate. The essential assumption of the Blasius type solution, namely the slow variation of velocity in the direction of the mean flow as compared to the direction normal to the mean flow, ceases to be valid in the neighborhood of the leading edge. The authors derive a solution of the Stokes type, i.e., neglecting the inertia terms, valid for the immediate neighborhood of the leading edge. This solution is then matched to the Blasius solution in two different ways. The range of validity of the potential, Stokes and Blasius solutions is illustrated.

H. W. Liepmann (Pasadena, Calif.).

Pinney, Edmund. Aerodynamic forces on a slotted flat plate. Quart. Appl. Math. 6, 81–83 (1948).

The derivation of the flat plate approximation to the aerodynamic forces exerted by an incompressible fluid on an airfoil is simple and is to be found in most textbooks. The present note treats the less simple case in which the airfoil contains slots oriented parallel to its length.

Extract from the paper.

Pröll, A. Ovale Strebenprofile in schiefer Anströmung. Österreich. Ing.-Arch. 2, 77–85 (1947).

In two-dimensional incompressible flow, it is known that flow patterns for oval bodies in a uniform stream can be constructed by using point source-sink combinations (when the uniform flow is taken in the direction of the line joining source to sink) or by using point vortex pairs (when the direction of the uniform flow is perpendicular to the line joining the vortices). Hence, as is proposed in this paper, the oblique flow about an oval body can be obtained by superimposing solutions for a "source-sink" body and for a "vortex" body if these bodies are sufficiently closely matched. The problem of constructing "vortex" bodies of various shapes and of matching a "vortex" body to a "source-sink" body is discussed in some detail. The solutions for the individual "source-sink" and "vortex" bodies are exact, but small errors occur in oblique flow in matching the body shapes. Numerical examples are included.

J. S. Isenberg (Providence, R. I.).

Kaplan, Carl. The flow of a compressible fluid past a curved surface. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 768, 23 pp. (1943).

The author solves the problem of compressible irrotational flow over a thin symmetrical body with cusps at the leading edge and the trailing edge. The free stream Mach number is less than one. The calculation uses the method of perturbation, i.e., the velocity potential is expanded into a power series of the thickness ratio. Terms up to third power of the thickness ratio are determined. As a check, the same problem is solved by the Rayleigh-Janzen method of power series in the free stream Mach number at small Mach numbers, and the results are compared. The velocity and the pressure distributions over the surface of the body are then computed and they show considerable deviation from the approximate theory of von Kármán when a local supersonic region appears. For purely subsonic flows, the von Kármán formula gives results which are in very close agreement with the author's more accurate calculation. The author concludes the paper by determining, for a given thickness ratio, the limiting free stream Mach number at which the power series in the thickness ratio probably diverge and thus the limiting Mach number is the probable limit of irrotational flow without shocks. *H. S. Tsien* (Cambridge, Mass.).

Kaplan, Carl. On similarity rules for transonic flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1527, 16 pp. (1948).

The author derives the similarity rules for transonic flows which were first announced by von Kármán [J. Aeronaut. Sci. 14, 373–402 (1947), p. 396; these Rev. 9, 111] and derived in detail by von Kármán himself [J. Math. Phys. Mass. Inst. Tech. 26, 182–190 (1947); these Rev. 9, 217]. The similarity parameter for the first appearance of sonic velocity is then calculated for two families of thin bodies studied previously by the author [cf. same Tech. Notes no. 1218 (1947); these Rev. 8, 541, and the preceding review]. *H. S. Tsien* (Cambridge, Mass.).

Robinson, A., and Young, A. D. Note on the application of the linearised theory for compressible flow to transonic speeds. Coll. Aeronaut. Cranfield. Rep. no. 2, 7 pp. (2 plates) (1947).

The authors show that, for finite aspect ratios of thin wings, the linearized theory of compressible flows gives finite and constant values for the slope of the lift coefficient against the angle of attack for all plan-forms when the Mach number approaches one from below. Moreover, for delta wings the same lift slope is obtained when the Mach number approaches one from above. Thus the linearized theory for three-dimensional wings is at least consistent in the transonic speeds, although the accuracy of this theory is expected to be poor for such speeds. *H. S. Tsien.*

Kahane, A., and Lees, Lester. The flow at the rear of a two-dimensional supersonic airfoil. J. Aeronaut. Sci. 15, 167–170 (1948).

The condition that must be satisfied immediately behind the trailing edge of a supersonic airfoil is that the regions of flow from the upper and lower surfaces, which have different entropies and are separated by a vortex sheet, have the same direction of flow and equal pressures. In this paper the flow conditions are determined by use of oblique-shock and isentropic-expansion equations, expressed as power series in the deflection angles. The angle of deflection at the trailing edge is found to vary as the fourth power of the incidence. Calculations are made for a flat plate and for thick airfoils. A qualitative discussion of the mechanism of return of the flow to the free-stream direction is included.

A previous paper on the same subject by Lighthill [Ministry of Aircraft Production [London] Aeronaut. Res. Committee, Rep. and Memoranda no. 1930(7412) (1944); these Rev. 7, 94] seems to be erroneous. *W. R. Sears.*

Falkovich, S. V. On the theory of a wing of finite span in a supersonic flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 391–394 (1947). (Russian. English summary)

In this paper a method for the determination of the velocity potential $\varphi(x, y, z)$ of a supersonic flow past a wing of finite span is described. Under the usual simplifying hypotheses this problem can be reduced to the determination of a solution of the equation $\varphi_{zz}+\varphi_{yy}-\varphi_{xx}=0$ for which, on the intersection of the wing with the (x, z)-axis (the sym-

metry axis of the wing), $(\partial\varphi/\partial y)_{y=0}$ is equal to a given function $\chi(x, z)$, and which vanishes on the envelope of the family of Mach cones with vortices along the leading edge. Using the formula of Volterra, the author obtains a representation for $\varphi(x, y, z)$ in terms of χ. The method is generalized for the case of a vibrating wing in a supersonic flow.

S. Bergman (Cambridge, Mass.).

Frankl', F. An investigation of the theory of a wing of infinite span moving at the speed of sound. Doklady Akad. Nauk SSSR (N.S.) **57**, 661–664 (1947).

The author shows that the function

$$(1)\qquad \psi=\sum_{2}^{\infty}[\beta_0 n^{\frac{1}{2}}+\beta_1+\beta_2 n^{-\frac{1}{2}}]z_{n/3}(q^2)\sin n\theta/z_{n/3}(q^{*2})$$

is a solution of the equations of Čaplygin representing the flow around a typical symmetric two-dimensional profile with the Mach number one at infinity. Here θ is the inclination of the velocity vector; q, its magnitude in terms of the velocity of efflux into a vacuum; q^*, the critical velocity; β_0, β_1 and β_2 are constants; $z_{n/3}(q^2)$ are asymptotic formulae for the functions of Čaplygin:

$$z_{n/3}(q^2)/z_{n/3}(q^{*2})=\lambda(n^{\frac{1}{2}}\eta)+n^{-\frac{1}{2}}\lambda^{(1)}(n^{\frac{1}{2}}\eta)+\cdots+n^{-\frac{1}{2}k}\lambda^{(k)}(n^{\frac{1}{2}}\eta)+n^{-\frac{1}{2}(k+1)}\Lambda_n^{(k+1)}(n^{\frac{1}{2}}\eta),$$

where η is a decreasing function of q equal to zero at q^*; λ is an Airy function defined by the conditions $d^2\lambda/d\xi^2-\xi\lambda=0$, $\lambda(0)=1$, $\lambda(\infty)=0$; $\lambda^{(i)}$ satisfy certain nonhomogeneous Airy equations and the conditions $\lambda^{(i)}(0)=\lambda^{(i)}(\infty)=0$; finally $|\Lambda_n^{(k+1)}(\xi)|<A^{(k+1)}$, $|\Lambda_n^{(k+1)'}(\xi)|<A^{(k+1)}$. A finite number of coefficients may be changed in equation (1) and ψ will still be a solution of the same problem. It is shown that Mach lines of these solutions cannot have double points outside a certain circle and that the sonic line and the Mach line merging with it at infinity are both asymptotic to the curve $y=\pm Cx^{5/4}$.

R. F. Clippinger.

Haskind, M. D. Oscillation of a wing in a subsonic gas flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 129–146 (1947). (Russian. English summary)

The determination of the oscillations of a wing in a subsonic two-dimensional gas flow can be reduced under the usual simplifying assumptions to the determination of the solution of the equation (A): $\varphi_{xx}+\varphi_{yy}+\nu^2\varphi=0$, satisfying the following boundary conditions on the axis, $y=0$: $\varphi_y=V(x)$, $|x|<1$; $\varphi=0$ for $x>1$; $\varphi_x-i\alpha\varphi=0$ for $x<-1$, where $V(x)$ is a known function, $\alpha=\lambda/M^2$, M being the Mach number at infinity, $\nu=\mu M/(1-M^2)$ the reduced frequency, λ and μ are constants. In order to solve the problem the author writes $\varphi=\varphi^{(0)}+\psi$, where $\varphi^{(0)}$ satisfies equation (A) and the boundary conditions $\varphi_y^{(0)}=V(x)$ for $|x|<1$, $y=0$; $\varphi^{(0)}=0$ for $|x|>1$, $y=0$. Finally it is required that in the neighborhood of $y=0$, $x=-1$, the following relation holds: $\nabla\varphi^{(0)}=C_1\delta_{\mp}^{-\frac{1}{2}}+\varphi_{01}$, $\delta_{\mp}^2=(x\pm1)^2+y^2$, where C_1 is a vector constant, and the vector function φ_{01} remains finite for $\delta_{\mp}\to0$. By conformal transformation $z=\cosh\zeta$, $\zeta=\xi+i\eta$, elliptic coordinates are introduced and the above equation assumes the form $\varphi_{\xi\xi}^{(0)}+\varphi_{\eta\eta}^{(0)}+\nu^2(\cosh^2\xi-\cos^2\eta)\varphi^{(0)}=0$. It can then be solved by separation of variables. The solutions of ordinary differential equations which we obtain in this manner can be represented by the Mathieu functions, Se_n, se_n. Thus one obtains for $\varphi^{(0)}$ the representation

$$\varphi^{(0)}=\sum_{n=1}^{\infty}a_n\,\mathrm{Se}_n(\xi)\,\mathrm{se}_n(\eta),$$

where the coefficient a_n can be written in the form of an infinite series.

The function ψ again satisfies equation (A) and for $y=0$, the boundary conditions $\psi=0$ for $x>1$, $\psi_y=0$ for $|x|<1$, and $\psi_x-i\alpha\psi=0$ for $x<-1$. Further, it becomes infinite in a prescribed manner at $x=\delta_{\mp}$. In order to determine ψ the author introduces a function W given by $\psi_x-i\alpha\psi=W_y$ and satisfying certain boundary conditions; he shows that $W(x, y)$ can be developed into a power series of (even) Mathieu functions, $W(\xi,\eta)=\sum_{n=0}^{\infty}b_n\mathrm{Ce}_n(\xi)\mathrm{ce}_n(\eta)$, where b_n again can be expressed in the form of an infinite series. He obtains $\psi(x, y)=e^{i\alpha x}\int_{+\infty}^{x}e^{-i\alpha u}(\partial W(u, y)/\partial y)du$. Applying the above results, he determines the lift and moment.

S. Bergman (Cambridge, Mass.).

Roberts, Richard Calvin. Note on the lift of a triangular wing at supersonic speeds. J. Math. Physics **27**, 49–55 (1948).

The well-known expression given by Stewart [Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109] for the lift of a triangular wing lying within the Mach cone is generalized for triangular wings of asymmetrical plan form.

E. N. Nilson (East Hartford, Conn.).

***Haack, W. Projectile shapes for smallest wave drag.** The Graduate Division of Applied Mathematics, Brown University. Translation no. **A9-T-3**. i+53 pp. 1948.

[Translation of Lilienthal Gesellschaft Bericht no. 139 (1941).] Using von Kármán's approximate theory of slender bodies in supersonic flow [Atti del V Convegno della "Fondazione Alessandro Volta," Rome, 1935, pp. 222–276], the author determines the optimum shapes of projectiles for prescribed (i) caliber and length, (ii) volume and length, (iii) volume and caliber. Only wave drag is considered. The method of calculation is first to attack the single variational problem of minimum drag for given caliber, length and volume, by standard methods. This is done both for the nose portion alone and for a complete projectile. There is some discussion of the fact that the optimal shapes obtained do not actually satisfy von Kármán's assumed conditions at the nose and tail, and also of other approximations of the theory. Some comparisons with experiment are included. [The reviewer noted an error in the formula for drag, for the case of given volume and length, in the table on page 39.]

W. R. Sears (Ithaca, N. Y.).

Evvard, John C. Theoretical distribution of lift on thin wings at supersonic speeds. (An extension). Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1585**, 49 pp. (1948).

Heaslet, Max. A., and Lomax, Harvard. The calculation of downwash behind supersonic wings with an application to triangular plan forms. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1620**, 43 pp. (1948).

Heaslet, Max. A., and Lomax, Harvard. Two-dimensional unsteady lift problems in supersonic flight. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1621**, 26 pp. (1948).

Cohen, Doris. The theoretical lift of flat swept-back wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1555**, iii+57 pp. (20 plates) (1948).

Kirkby, S., and Robinson, A. Wing body interference at supersonic speeds. Coll. Aeronaut. Cranfield. Rep. no. 7, 12 pp. (5 plates) (1947).

Robinson, A., and Hunter-Tod, J. H. Bound and trailing vortices in the linearised theory of supersonic flow, and the downwash in the wake of a delta wing. Coll. Aeronaut. Cranfield. Rep. no. 10, 13 pp. (4 plates) (1947).

Robinson, A., and Hunter-Tod, J. H. The aerodynamic derivatives with respect to sideslip for a delta wing with small dihedral at supersonic speeds. Coll. Aeronaut. Cranfield. Rep. no. 12, 20 pp. (4 plates) (1947).

Carafoli, E. Influence du virage circulaire plan sur les propriétés aérodynamiques des ailes. Acad. Roum. Bull. Sect. Sci. 27, 45–50 (1947).

Stupočenko, E. V. The influence of external friction on the formation of shock waves in cylindrical tubes. Doklady Akad. Nauk SSSR (N.S.) 59, 1073–1076 (1948). (Russian)

This paper deals with the influence of the wall friction on the formation of shock waves in cylindrical tubes in which an elastic fluid is set in motion by a piston, by generalizing the method of H. Hugoniot [J. École Polytech. 57, 3–97 (1887); 58, 1–125 (1889)] for the frictionless case.

A. W. Boldyreff (Albuquerque, N. M.).

Paterson, Stewart. The compression of a gas by a rigid piston moving in a closed rigid tube. Philos. Mag. (7) 39, 76–82 (1948).

Given a column of ideal gas confined at rest within a rigid cylinder, suppose that a piston starts to move with constant velocity into the cylinder. A shock wave S_1 is generated in the gas and is reflected back and forth as shock waves $S_2, S_3, \cdots$. The author develops expressions for the gas pressure, volume and temperature behind S_n, and gives approximations for the cases of n small and large.

C. C. Torrance (Annapolis, Md.).

Stoker, J. J. The formation of breakers and bores. The theory of nonlinear wave propagation in shallow water and open channels. Communications on Appl. Math. 1, 1–87 (1948).

This is an exhaustive treatise on the two-dimensional nonsteady flows of water in open channels. As shown by K. O. Friedrichs in the appendix, in the case where the product of the undisturbed water depth and the curvature of the free surface is small the problem can be analyzed by the so-called nonlinear shallow water theory which is based upon a set of differential equations analogous to those of gas dynamics. The method of characteristics for solving the gas-dynamical problems, intensively developed in recent years, can then be brought over to the channel problems. The only new element is in connection with the formation of the mathematical discontinuities in the solution, called shocks in accordance with the terminology of gas dynamics. These shocks are identified with the breakers and bores. Their mathematical determination is made unique in the present case by the condition that they always induce a loss of the total energy. This loss of energy is of course transformed into turbulence and eddies and finally into heat.

The case when a disturbance at one point in the water propagates into still water of constant depth is particularly simple as one of the two families of characteristics consists entirely of straight lines. This is called the case of simple waves. It is shown that the propagation of a hump always leads to shock while the propagation of a depression eventually dies out. These concepts are applied to the problems of flow after the breaking of a dam, flood waves in rivers and the breaking of waves. The accompanying study of shocks gives an analysis of bores, hydraulic jumps and the reflection of bores from a rigid wall.

The paper concludes with a few examples of numerical calculation in the more complicated case of the breaking of waves on a uniformly sloping beach. Here the distance of the breaking point from the origin of the wave, which started as a sine wave, is determined by the two parameters: the ratio of the amplitude of the wave at the start to the water depth at the origin and the ratio of the wave length to the distance from the origin to the shore line. However, no general rules can be given as the number of cases treated is too small.

H. S. Tsien (Cambridge, Mass.).

Platzman, George W. The partition of energy in periodic irrotational waves on the surface of deep water. J. Marine Research 6, 194–202 (1947).

Evangelisti, Giuseppe. Sulla propagazione delle piccole onde nei canali a sezione variabile. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) 9, 111–121 (1942).

Brehovskih, L. On the limits of applicability of certain methods of approximation used in acoustics. Doklady Akad. Nauk SSSR (N.S.) 58, 587–590 (1947). (Russian)

The representation of the acoustical field in closed buildings as a superposition of the direct spherical wave and an infinite number of other spherical waves, reflected from the walls, is only approximate. For an infinite plane wall this method of "virtual sources" is found to be rigorously applicable only when the reflection coefficient is independent of the angle of incidence. When the reflecting zones are not very large in comparison with the wave length, one must also take diffraction into account. A criterion is given for judging whether geometrical acoustics still holds approximately or not. Another question is that of the limits within which the use of the normal value of the impedance is allowed. For a plane wave of the type $\varphi = \varphi_0 \exp(-i\omega t + iks)$ with a finite angle of incidence on a plane surface separating two media, one finds the condition $k^2 \ll |k_1|^2$, where the subscript 1 applies to the second medium.

G. Toraldo di Francia (Florence).

Hostinský, Bohuslav. On mean values of the energy of air oscillating in a box. Rozpravy II. Třídy České Akad. 52, no. 2, 19 pp. (1942). (Czech)

A German summary appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 43, 24–36 (1942); these Rev. 9, 118.

Elasticity, Plasticity

Zanaboni, Osvaldo. I solidi semielastici. Proposta di una teoria generale indipendente dal legame sforzi-deformazioni. Rend. Sem. Mat. Fis. Milano 15, 1–28 (1941).

In this paper a semielastic solid means a continuous medium in which there may exist stress and infinitesimal displacement (and consequently strain), but neither a strain-energy function nor an explicit law connecting stress

and strain is assumed. On the basis of the principle of virtual work the author claims to establish two theorems which extend to semielastic solids the two basic minimal principles of elasticity. A section is devoted to the principle of Saint Venant. The whole paper is purposely written with a minimum of symbolism and consequently is difficult to follow as a mathematical argument. The meaning of the two theorems referred to above, and their proofs, are not clear to the reviewer. *J. L. Synge* (Dublin).

Zanaboni, Osvaldo. Equazioni di equilibrio e di congruenza degli archi semielastici a doppia curvatura. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **1**, 33–46 (1944).

Bažant, Zdeněk. The theory of the elastic half-space. Rozpravy II. Třídy České Akad. **53**, no. 20, 24 pp. (1943). (Czech)

A German summary appeared in Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 313–329 (1943); these Rev. **8**, 546.

Hruban, K. Deformation and stress in a heterogeneous half-space. Rozpravy II. Třídy České Akad. **55**, no. 13, 23 pp. (1945). (Czech)

Diaz, J. B., and Weinstein, A. The torsional rigidity and variational methods. Amer. J. Math. **70**, 107–116 (1948).

The starting point for the estimation of the torsional rigidity S is the formula $S=P-D(\psi)$ (P, polar moment of inertia; ψ, conjugate function of the warping function; $D(\psi)$, Dirichlet's integral). The authors get an upper bound for $D(\psi)$ by the Rayleigh-Ritz method and a lower bound by using Schwarz's inequality and a transformation involving Green's formula and derive as a simple application an inequality for S for a quadratic frame.

In the second part the authors show that for

$$I(u)=\int\int (u_x^2+u_y^2-4u)dxdy=\text{minimum}$$

the equalities

$$D(\psi_N^*)=-I(\psi_N^*),\quad 2\int\int \psi_N^* dxdy=-I(\psi_N^*)$$

hold, provided that $\psi_*^*=\sum c_i f_i(x,y)$ is the best approximation to ψ in the sense of the Rayleigh-Ritz method obtained by using N fixed functions $f_i(x,y)$, $i=1,2,\cdots,N$, satisfying the boundary condition. *P. Funk* (Vienna).

Hemp, W. S. On the analysis of statically indeterminate structures. Coll. Aeronaut. Cranfield. Rep. no. **3**, 8 pp. (1946).

Bolle, L. Contribution au problème linéaire de flexion d'une plaque élastique. Bull. Tech. Suisse Romande **73**, 281–285, 293–298 (1947).

The author treats the effect of transverse shear deformation on the bending of elastic plates in the following manner. He modifies Navier's hypothesis, according to which the normal to the undeformed middle surface is deformed without extension into the normal to the deformed middle surface, by postulating merely that the normal to the undeformed middle surface is deformed without extension into another straight line. The inclination of this straight line, with reference to the slopes of the deformed middle surface, is determined approximately by suitably averaging over the thickness the relations between transverse shearing stress and transverse shearing strain. The author does not actually carry out this averaging process but rather assumes that known corresponding results for beams of rectangular cross section may be taken over into plate theory. In this way a system of differential equations of the sixth order is obtained which is similar to, and in the case of a plate acted upon by edge loads only is equivalent to, results previously obtained in a different manner by the reviewer [J. Math. Physics **23**, 184–191 (1944); J. Appl. Mech. **12**, A-69–A-77 (1945); these Rev. **6**, 195; **7**, 42]. As applications of the theory the author solves several specific problems dealing with circular plates and compares the results with the results of the theory in which transverse shear deformation is not taken into account.

An important consequence of taking into account transverse shear deformation in plate theory is the result found by the reviewer and once more in this paper that thereby it becomes possible and necessary to satisfy three boundary conditions along the edge of the plate, instead of the two conditions which Kirchhoff has shown to be appropriate when the theory is developed on the basis of Navier's hypothesis. In this connection the author points out that a plate theory with three boundary conditions had already been proposed by M. Levy [J. Math. Pures Appl. (3) **3**, 219–306 (1877)]. (Levy's method consisted in superimposing on the Navier-Kirchhoff solution a particular solution of the equations of the three-dimensional theory, of the form $u=\partial F/\partial y$, $v=-\partial F/\partial x$, $w=0$, where x, y are coordinates in the plane of the plate, and where F is a solution of $\nabla^2 F=0$ satisfying the condition $\partial F/\partial z=0$ on the surfaces $z=\pm h/2$ of the plate. This additional solution can be taken in the form $F=\zeta(x,y)\sin \pi z/h$ and the function ζ is to be determined such that the combined solution satisfies three boundary conditions along the edge of the plate.) The author remarks that Levy's theory was attacked as unsound by Boussinesq [cf. C. R. Acad. Sci. Paris **86**, 461–463 (1878)] and that Saint-Venant in his annotated edition of Clebsch's book on elasticity [Théorie de l'Élasticité des Corps Solides, Paris, 1883] effectively failed to see the significance of Levy's work. [There is no mention of Levy's paper in Love's A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927, which devotes considerable space to the problem in question, on pp. 27–29 and pp. 458–463.] It is interesting to note that Levy's paper also contains the exact solution of the problem of the moderately thick plate acted upon by forces applied at its edge only, which in Love's Treatise [pp. 465–474] is credited to J. H. Michell. The author correctly contends that Boussinesq's objections to Levy's theory were unfounded. *E. Reissner* (Cambridge, Mass.).

Supino, G. Sopra la teoria delle lastre elastiche. Ann. Mat. Pura Appl. (4) **24**, 39–64 (1945).

This paper is concerned with the theory of transverse bending of moderately thick plates. After pointing out that the Michell solutions of this problem, in common with the corresponding theory of thin plates, admit satisfaction of two boundary conditions only instead of three conditions the author shows that superposition of a special system of solutions of the three-dimensional equations of the theory of elasticity permits satisfaction of three boundary conditions. The special system of solutions is essentially that given by M. Levy [see the preceding review]. The author

discusses the applications of this idea, which is here ascribed to Almansi [see Supino, same Ann. (4) 17, 307–326 (1938)].
E. Reissner (Cambridge, Mass.).

Buell, E. L. On the distribution of plane stress in a semi-infinite plate with partially stiffened edge. J. Math. Physics 26, 223–233 (1948).

A semi-infinite plate in plane stress occupies the positive half plane $y>0$. One edge $y=0$ has a stiffener $(0<x<+\infty)$ to which a load P is applied at $x=0$ and parallel to the stiffened edge. An Airy's stress function, $\varphi=y\varphi_1+\varphi_2$, where φ_1 and φ_2 are potential functions, is determined to satisfy the mixed boundary conditions on $y=0$ by conformally mapping the plate region within the unit circle of the ζ plane. For the latter domain a solution is obtained in the form of a Fourier series whose coefficients satisfy an infinite system of linear equations in an infinite number of unknowns. This system is solved approximately for the first few coefficients. Stresses are calculated along the edge of the plate and compared with the results of an unstiffened edge or with an edge stiffened on $-\infty<x<+\infty$ and the "effective length" is determined. *D. L. Holl* (Ames, Iowa).

Chien, Wei-Zang. Large deflection of a circular clamped plate under uniform pressure. Chinese J. Phys. 7, 102–113 (1947). (English. Chinese summary)

The problem is studied by the method of successive approximations in terms of a parameter representing the ratio of the center deflection to the plate thickness. The author employs von Kármán's nonlinear system of equilibrium equations. By expanding the deflection, the radial and circumferential stresses in ascending power series a sequence of linearized problems is solved, the first approximation yielding the results of the first order thin plate theory of Poisson and Kirchhoff. Additional approximations for the deflection and planar stretching forces are obtained and the results compared with those of Way, Federhofer and Nadai with good agreement. A study of the von Mises-Hencky theory of failure is made for the condition of yielding at the clamped edge. *D. L. Holl* (Ames, Iowa).

Ling, Chih-Bing. On the stresses in a notched plate under tension. J. Math. Physics 26, 284–289 (1948).

The plane-stress distribution associated with a semi-infinite plate whose edge contains a circular arc notch (not necessarily semi-circular) is investigated for the case where, far from the notch, a state of uniform tension exists. The stress function is found in terms of integrals of known biharmonic functions expressed in bipolar coordinates. An explicit integral formula is found for the "stress concentration factor" and this result is plotted for a complete range of notch depths including "negative notches" (mounds). It is stated that the stress function used can be applied to certain other plate problems. *G. F. Carrier.*

Conway, H. D. The bending of symmetrically loaded circular plates of variable thickness. J. Appl. Mech. 15, 1–6 (1948).

An investigation devoted to a study of deflections in thin isotropic circular plates, whose thickness varies linearly with the distance from the center, and which are subjected to a symmetric distribution of normal load. The paper contains calculations for the maximum stresses and displacements in six special cases. *I. S. Sokolnikoff.*

Budiansky, Bernard, Hu, Pai C., and Connor, Robert W. Notes on the Lagrangian multiplier method in elastic-stability analysis. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1558, 46 pp. (1948).

New applications of the Lagrangian multiplier method to stability analysis are described by means of elementary examples. A detailed analysis for finding upper and lower limits to critical stresses of clamped rectangular plates is presented in an appendix. *From the authors' summary.*

Budiansky, Bernard, and Connor, Robert W. Buckling stresses of clamped rectangular flat plates in shear. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1559, 11 pp. (1948).

By consideration of antisymmetrical, as well as symmetrical, buckling configurations, the theoretical shear buckling stresses of clamped rectangular flat plates are evaluated more correctly than in previous work. The results given, which represent the average of upper- and lower-limit solutions obtained by the Lagrangian multiplier method, are within $1\frac{1}{4}$ percent of the true buckling stresses.
Authors' summary.

Budiansky, Bernard, Connor, Robert W., and Stein, Manuel. Buckling in shear of continuous flat plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1565, 24 pp. (1948).

Hopkins, H. G. The solution of small displacement, stability or vibration problems concerning a flat rectangular panel when the edges are either clamped or simply supported. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2234 (8894), 18 pp. (1945).

Džanelidze, G. Yu. Survey of the work published in the USSR on the theory of the bending of thick and thin plates. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 109–~~125~~ 128 (1948). (Russian)

Mandel, Jean. Sur la détermination du centre de torsion d'un cylindre à l'aide du théorème de réciprocité. C. R. Acad. Sci. Paris 226, 779–781 (1948).

The author gives the following results for the center (ξ, η) of torsion of a cantilever beam under the combined action of flexure and torsion produced by an applied end load:

$$\xi I_x=-\int\int_{\Sigma} y\varphi(x,y)dA+k\int\int_{S} x\Psi(x,y)dA,$$

$$\eta I_y=\int\int_{\Sigma} x\varphi(x,y)dA+k\int\int_{S} y\Psi(x,y)dA.$$

Here I_x, I_y are principal moments of inertia of the section; φ, ψ are harmonic conjugate functions, with φ the Saint Venant warping or torsion function; $\Psi=\psi-\frac{1}{2}(x^2+y^2)$, $k=\sigma/(1+\sigma)$, where σ is Poisson's ratio; S denotes the entire cross section enclosed by the exterior boundary and Σ the net section after deleting the interior openings. The results are obtained by using Betti's reciprocal theorem for sets of forces and displacements which accompany pure flexure and pure torsion as given by Saint Venant's theory. The results are known when the torsion problem is solved and depend on the form of the cross section and Poisson's ratio. Weinstein and Trefftz regard the center of shear as independent

of Poisson's ratio. If $k=0$, the above results agree with Weinstein [Quart. Appl. Math. **5**, 97–99 (1947); these Rev. **8**, 546] and Trefftz. *D. L. Holl* (Ames, Iowa).

Smith, C. B., and Voss, A. W. Stress distribution in a beam of orthotropic material subjected to a concentrated load. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1486**, 37 pp. (1948).

Houbolt, John C., and Anderson, Roger A. Calculation of uncoupled modes and frequencies in bending or torsion of nonuniform beams. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1522**, 75 pp. (1948).

The primary purpose of this paper is to provide the vibration analyst with a numerical procedure that yields the frequencies of fundamental and higher modes with practical accuracy by means of elementary computations closely related to those involved in engineering beam theory. The procedures presented, based on the Stodola iteration method, require only tabular computations that may be performed mentally or by slide rule. The applicability of the procedure to a wide variety of beams (e.g., cantilever beams with concentrated mass, beams mounted on springs, etc.) is illustrated by means of many examples performed in detail.

B. Budiansky (Providence, R. I.).

Anderson, Roger A., and Houbolt, John C. Effect of shear lag on bending vibration of box beams. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1583**, 15 pp. (1948).

Mikeladze, Š. E. Longitudinal-transverse bending of a beam on an elastic foundation. Doklady Akad. Nauk SSSR (N.S.) **59**, 451–454 (1948). (Russian)

The equation of the central line of an elastic beam, resting on a yielding foundation and subjected to a constant compressive load and varying transverse load, is obtained in the form of a rapidly converging trigonometric series.

I. S. Sokolnikoff (Los Angeles, Calif.).

Swainger, K. H. A new criterion of yielding in metals due to complex stresses. Philos. Mag. (7) **39**, 122–133 (1948).

For combined tension and torsion of thin-walled tubes, the yield condition of Mohr demands that $\sigma^2+4\tau^2$ be constant, and that of von Mises that $\sigma^2+3\tau^2$ be constant, where σ denotes the tensile stress and τ the shearing stress. The author therefore suggests the use of $\sigma^2+c\tau^2=$constant as yield condition, where the coefficient c is to be determined by experiment. This condition is then generalized to triaxial stress. The resulting formula is not very convenient for practical purposes because it requires the stress tensor to be referred to a particular set of rectangular axes.

W. Prager (Providence, R. I.).

Girkmann, K. Die Beanspruchung einer Druckschachtpanzerung bei unvollständiger Umschliessung. Österreich. Ing.-Arch. **2**, 211–225 (1948).

The paper is concerned with the stresses in a thin-walled elastic tube which is embedded in an elastic body and subjected to interior pressure. The tube is assumed to be in a state of plane strain, and the surrounding elastic material is assumed to resist the radial or circumferential displacement of a tube element with surface stresses which are proportional to these displacement components, different factors of proportionality being used for the radial and circumferential directions. The author discusses, in particular, the stresses occurring when the elastic support extends only over part of the tube circumference (longitudinal cavity between tube and surrounding material). It is shown that these stresses may considerably exceed those in a tube which has no elastic support at all but is allowed to deform with axial symmetry. *W. Prager* (Providence, R. I.).

Moufang, Ruth. Strenge Berechnung der Eigenspannungen, die in plastisch aufgeweiteten Hohlzylindern nach der Entlastung zurückbleiben. Z. Angew. Math. Mech. **28**, 33–42 (1948).

The paper contains an analysis of the residual stresses in a thick-walled tube which has been plastically deformed by interior pressure. This analysis is based on the strain-hardening law of R. Schmidt and on the assumption of plane strain. *W. Prager* (Providence, R. I.).

MacGregor, C. W., Coffin, L. F., Jr., and Fisher, J. C. The plastic flow of thick-walled tubes with large strains. J. Appl. Phys. **19**, 291–297 (1948).

The paper presents an analysis of the plastic flow of thick-walled tubes under exterior and interior pressures and axial load. Finite strains are considered and the discussion is based on a stress-strain law which is patterned on Nadai's law for strain-hardening materials but uses the so-called logarithmic strain instead of the conventional (infinitesimal) strain. The theoretical results are compared with experimental data concerning the dependence of the circumferential strains at the cylindrical surfaces of an aluminum tube on the interior pressure. The predictions of the present theory are represented by a curve which lies between the empirical curve and the curve representing the predictions of the infinitesimal strain theory. *W. Prager.*

Panov, D. Yu. On the stability of bimetallic shells under heating. (On the theory of a thermal switch.) Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 603–610 (1947). (Russian)

The paper is concerned with the investigation of stability of a shallow bimetallic shell used in thermostat switches.

I. S. Sokolnikoff (Los Angeles, Calif.).

Zvolinskiĭ, N. The propagation of a disturbance caused by a point impulse in an elastic half space covered by a layer of fluid. Doklady Akad. Nauk SSSR (N.S.) **59**, 1081–1084 (1948). (Russian)

A solid elastic half-space $y>0$ is covered by a layer of elastic liquid of depth h. The surface of contact is the (x, z)-plane. At the instant $t=0$ a cylindrical impulse of dilatational type is produced along the line $x=0$, $y=y_0>0$. The pressure on the surface of the liquid is constant. On the interface between liquid and solid tangential stresses vanish and normal stresses and velocities are continuous. The state of the system for $t>0$ is determined by constructing the solution with the aid of a certain family of plane waves. The related problem involving a concentrated impulse of rotational type can also be solved by the same method. *I. S. Sokolnikoff* (Los Angeles, Calif.).